LINEAR DIFFERENTIAL EQUATIONS OF INFINITE ORDER*

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1. *Introduction.* In this address we shall give such a conspectus of the theory of linear differential equations of infinite order as will enable the reader rapidly to orient himself with respect to this subject. Neither in the presentation of results already developed nor in the bibliographical references (in the footnotes) is there any attempt to attain an exhaustive account. The purpose is rather that of a general outlook on the subject such as may interest a considerable number of persons and may serve as a point of departure for a few investigators who may desire to penetrate a relatively new and unexplored domain, the importance of which will certainly be more fully recognized as the subject is further developed in the next two or three decades.

In §2 the general nature of the problem of linear differential equations of infinite order is indicated partly by means of examples and partly by means of notions of a general character which are intimately associated with the somewhat more special problem which is the center of our present interest. In §3 are set forth some of the many connections of this problem with other matters of more or less wide interest in the field of analysis. A brief account of the present state of knowledge with reference to linear differential equations of infinite order with constant coefficients is given in §4, while §§5 and 6 treat (but with somewhat less fullness) the corresponding matters for equations with polynomial and with analytic coefficients, respectively. Finally, §7 is devoted to a brief account of some further problems and connections of the theory of differential equations of infinite order.

2. *Nature of the Problem.* A typical problem in the theory of linear differential equations of infinite order is that of solving the equation

$$(1) \qquad a_0(x)y + a_1(x)y' + a_2(x)y'' + \cdots = \phi(x),$$

* Retiring address of the Chairman of Section A of the A.A.A.S., delivered at St. Louis, December 31, 1935.

where $\phi(x)$, $a_0(x)$, $a_1(x)$, $\cdots$ are given functions of x subject to suitable conditions and where y is the function to be determined, the primes denoting differentiation with respect to x. In the investigation of such equations the unknown function $y(x)$ is often required to belong to a particular class of functions such, for instance, as the functions of exponential type.

One may also investigate systems such as the following:

$$(2) \qquad \frac{dy_i}{dx} = \sum_{j=1}^{\infty} A_{ij}(x)y_j(x) + \Phi_i(x), \qquad (i = 1, 2, 3, \cdots),$$

where the A's and the Φ's are given functions and the y's are the functions to be determined.

In the infinite case the analogy between (1) and (2) does not seem* to be as close as that between the corresponding equation and system in the finite case. Thus, if we consider the special system

$$\alpha_0(x)\frac{dy_1}{dx} + \alpha_1(x)y_1 + \alpha_2(x)y_2 + \cdots + \alpha_n(x)y_n + \cdots = 0,$$

$$\frac{dy_2}{dx} = y_1, \quad \frac{dy_3}{dx} = y_2, \cdots, \quad \frac{dy_n}{dx} = y_{n-1}, \cdots,$$

and if we write y for y_n, then the functional equation for y may be indicated by the following relation which involves a limiting process as to n:

$$(3) \qquad [\alpha_0(x)y^{(n)} + \alpha_1(x)y^{(n-1)} + \cdots + \alpha_{n-1}(x)y' + \alpha_n(x)y]_{n=\infty} = 0.$$

There is a marked difference between the theories of (1) and (3).

Systems of equations of the form

$$(4) \qquad \sum_{k=1}^{m}\sum_{\nu=0}^{\infty} a_{ik\nu}(x)y_k^{(\nu)} = \phi_i(x), \qquad (i = 1, 2, \cdots, m),$$

have been considered both when m is finite and when m is infinite. Systems (2) are capable of obvious generalizations.

In a few instances linear partial equations of infinite order have appeared. Non-linear ordinary equations of infinite order have also been investigated to some extent.

* See T. Lalesco, Journal de Mathématiques, (6), vol. 4 (1908), pp. 125–202, especially pp. 182–183.

For the sake of unity we shall direct our attention mainly (but not entirely) to (1) and (4), particularly since by means of them we shall be able to set forth that part of the theory which up to the present has received the major share of attention.

Two pioneers in the development of the theory of linear differential equations of infinite order were S. Pincherle* and C. Bourlet,† the priority belonging to the former. But the work of Bourlet was independent of that of Pincherle. In fact, the two authors approached the problem from quite different points of view; but their results overlapped in important ways. Recently P. Flamant‡ has further investigated the problem from the point of view of Bourlet and Pincherle. Bourlet used the term *transmutation* to denote an operation T which makes a given function $\phi(x)$, the object of the operation, correspond to another function $T\phi(x)$, the result of the operation. The transmutation T is said to be distributive if for arbitrary functions $\phi(x)$ and $\psi(x)$ and for an arbitrary constant c, we have

$$T[\phi(x) + \psi(x)] = T\phi(x) + T\psi(x), \quad T[c\phi(x)] = cT\phi(x).$$

Pincherle simply calls T a distributive operation. Bourlet introduces at once a certain notion of continuity with respect to the transmutation, but his account of the matter is lacking in clarity. Both Bourlet and Pincherle insist on the proposition that every additive, uniform, continuous, and "regular" transmutation can be represented by a series of the form

$$Tu = \sum_{n=0}^{\infty} a_n(x) \frac{d^n u}{dx^n},$$

thus bringing the theory of transmutations into intimate association with the theory of differential equations of infinite order. Flamant (loc. cit.) has undertaken to subject transmutations to a more rigorous analysis than that which satisfied the earlier

* For Pincherle's contributions to this problem see the outline of his work in Acta Mathematica, vol. 46 (1925), pp. 341–362.

† For Bourlet's work, see especially Annales de l'École Normale Supérieur, (3), vol. 14 (1897), pp. 133–190; (3), vol. 16 (1899), pp. 333–375; and the papers of Flamant mentioned in the next note.

‡ For papers by Flamant, see especially Palermo Rendiconti, vol. 48 (1924), pp. 135–208; Bulletin des Sciences Mathématiques, (2), vol. 52 (1928), pp. 26–48, 77–96, 104–128; Palermo Rendiconti, vol. 54 (1930), pp. 371–413.

investigators and in particular to clarify the situation as regards the proposition just quoted by giving precise and sufficiently general conditions for its validity. He also generalizes the formula by replacing the operation D of differentiation by certain more general operations L.

The particular transmutation Tu,

$$Tu = u - \frac{x}{1!}u' + \frac{x^2}{2!}u'' + \cdots + (-1)^n \frac{x^n}{n!}u^{(n)} + \cdots,$$

will bring to notice (see Bourlet, loc. cit., 1897, pp. 174–176) one aspect of the difference between equations of finite order and those of infinite order. For functions analytic at $x=0$ one has $Tu=u(0)$. Hence this transmutation makes every such function correspond to a constant. The inversion of the transmutation is therefore generally impossible in the domain of functions analytic at $x=0$; it is only constants which admit inverses. If a denotes an arbitrary constant, then the equation $Tu=a$ is a linear differential equation of infinite order which is verified by the function $u=a+xf(x)$, where $f(x)$ is an *arbitrary* function analytic at $x=0$.

A much more instructive example is the following one:*

$$(1-x)y + y' + \cdots + \frac{y^{(n)}}{n!} + \cdots = 0.$$

The classic function $\Gamma(x)$ satisfies this equation in every region of the plane exterior to the circles of radius 1 about the points $0, -1, -2, \cdots, -n, \cdots$ as centers, since for such regions the equation (for suitable functions) reduces to the difference equation $y(x+1)-xy(x)=0$. But, in a point situated in the interior of one of the preceding circles, the development $\sum y^{(n)}(x)/n!$ is evidently divergent for $y=\Gamma(x)$, although the product $x\Gamma(x)$ is everywhere finite in one of these circles save at the center. Therefore the given differential equation is not satisfied by $y=\Gamma(x)$ in the interior of one of these circles. This circumstance is new and is characteristic of linear differential equations of infinite order. It is therefore not sufficient to be assured that a non-integral analytic function satisfies a linear differential equation of infinite order in a point in order to conclude

* See Lalesco, loc. cit., 1908, pp. 194–195.

that it satisfies the equation throughout the domain of its existence. Lalesco (loc. cit.) investigates the problem which is set by the existence of such situations as this example brings to light. The reader is referred to his memoir for the results.

3. *Connections of the Problem.* Pincherle,* in one of his early papers, has indicated certain interesting connections of the problem of linear differential equations of infinite order. Let $A(z)$ denote the function

$$A(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}},$$

which is regular outside of the circle of radius R about 0 as center. Put $z = y - x$. Then the function $A(y-x)$ is regular when $|y-x| > R$, and hence when either $|y| > |x| + R$ or $|x| > |y| + R$. Under the first of the last two hypotheses, if we take x interior to a circle of radius σ about 0 as center and y exterior to the circle of radius $R+\sigma$ about 0 as center, then we may write $A(y-x)$ in either of the forms

$$A(y - x) = \sum_{n=0}^{\infty} \frac{a_n}{(y - x)^{n+1}}, \tag{5}$$

$$A(y - x) = \sum_{n=0}^{\infty} \frac{A_n(x)}{y^{n+1}}, \tag{6}$$

where the $A_n(x)$ constitute the system of Appell polynomials with the coefficients a_n (and satisfying the relation $A_n'(x) = nA_{n-1}(x)$). Under the second of the hypotheses, if we take y interior to the named circle of radius σ, and x exterior to the named circle of radius $R+\sigma$, then we have for $A(y-x)$ either the development (5) or the development

$$A(y - x) = \sum_{n=0}^{\infty} \frac{y^n}{n!} A^{(n)}(- x), \tag{7}$$

where $A^{(n)}(z)$ is the nth derivative of $A(z)$.

One now considers the expression

* Pincherle, Memorie Reale Accademia delle Scienze, Bologna, (4), vol. 9 (1888), pp. 45–71; French translation in Acta Mathematica, vol. 48 (1926) pp. 279–304.

$$A(\psi) = \int_{(l)} A(y - x)\psi(y)dy, \tag{8}$$

where the integration is taken along a line (l) of the y plane, and $\psi(y)$ is an analytic function without singularities along (l). One distinguishes two cases as follows:

(a) The line (l) (either finite in length or infinite under suitable conditions) is such that the modulus of each point on it exceeds $R+\sigma$. Then, for $|x| < \sigma$, one can admit for $A(y-x)$ either of the developments (5) and (6). On putting

$$\int_{(l)} \frac{\psi(y)dy}{y - x} = \phi(x), \tag{9}$$

one has from (8), in the respective cases, the developments

$$A(\psi) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \phi^{(n)}(x), \tag{10}$$

$$A(\psi) = \sum_{n=0}^{\infty} c_n A_n(x), \qquad c_n = \int_{(l)} \frac{\psi(y)dy}{y^{n+1}}. \tag{11}$$

(b) The line (l) is such that all its points are interior to the circle of radius σ about 0 as center. Suppose that $|x| > R+\sigma$. For $A(y-x)$ we may then use either of the developments (5) and (7). Then from (9) we have, in the respective cases, the relations

$$A(\psi) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \phi^{(n)}(x), \tag{12}$$

$$A(\psi) = \sum_{n=0}^{\infty} c_n' \frac{A^{(n)}(-x)}{n!}, \qquad c_n' = \int_{(l)} y^n \psi(y)dy. \tag{13}$$

Now let $f(x)$ be a given function and consider the functional equation

$$A(\psi) = f(x). \tag{14}$$

The solution of this equation involves the inversion of the integral in (8). The formulas which have been given show that this problem, at least in many of its essential elements, coincides with other functional problems as follows:

(a) to solve a linear differential equation of infinite order with constant coefficients, namely, the equation

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} \phi^{(n)}(x) = f(x);$$

(b) to find the development of a given function of x in terms of a given system of Appell polynomials;

(c) to develop a given function $f(x)$ in a series in terms of the successive derivatives of a given function $A(-x)$.

In the paper here cited Pincherle makes an important contribution toward the solution of these problems. He has also treated them and other closely related matters in many other memoirs whose dates of publication now stretch over almost the whole of a half-century period.

H. T. Davis* has given a method for deriving the Fredholm theory from the theory of differential equations of infinite order.

I. M. Sheffer† and others have indicated the close connection which exists between the theory of a system of infinitely many linear equations in infinitely many unknowns and the theory of linear differential equations of infinite order. To pass from the latter to the former, when working in the field of analytic functions, one has only to assume a power series expansion for the unknown function, substitute into the differential equation, and equate coefficients. Conversely, if one is given the system of equations

$$\sum_{j=0}^{\infty} a_{ij} y_j = c_i, \qquad (i = 0, 1, \cdots),$$

where the a_{ij} and the c_i are wholly arbitrary, then there exists a linear differential equation

$$P_0(x)y(x) + P_1(x)y'(x) + \cdots + P_n(x)y^{(n)}(x) + \cdots = C(x),$$

where the expressions

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* H. T. Davis, Annals of Mathematics, vol. 28 (1927), pp. 309–317.

† I. M. Sheffer, Annals of Mathematics, vol. 30 (1929), pp. 250–264. See also, among others, E. Hilb, Mathematische Annalen, vol. 82 (1920), pp. 1–39, vol. 84 (1921), pp. 16–30, 43–52, vol. 85 (1922), pp. 88–89; O. Perron, Mathematische Annalen, vol. 84 (1921), pp. 1–15, 31–42; F. Lettenmeyer, Dissertation, München, 1927, pp. 1–52.

$$C(x) = \sum_{k=0}^{\infty} c_k x^k \qquad \text{and} \qquad P_i(x) = \sum_{k=0}^{\infty} p_{ik} x^k$$

are formal power series in x such that on setting $y(x) = \sum y_k x^k$ and substituting into the differential equation and equating coefficients the resulting system is identical with the given one; and it is easy to obtain the relations between the p's and the a's to effect this formal equivalence.

D. C. Lewis* developed the theory of infinite systems of ordinary non-linear differential systems of infinite order with applications to certain second order partial differential equations, using for this purpose certain integral forms which are equivalent to the given systems with specified initial conditions.

In a paper on the Laplace differential equation of infinite order, namely, the equation

$$\sum_{n=0}^{\infty} (a_{n0} + a_{n1}x + a_{n2}x^2 + \cdots + a_{np}x^p)u^{(n)}(x) = f(x),$$

where p is a positive integer and not all the quantities a_{np} are zero, H. T. Davis† pointed out that the general theory of this equation formally unifies the theories of the following types of functional equations where the $p_i(x)$ are polynomials of degree not greater than p:

(a) $$u(x) + \int_x^{\infty} \sum_{i=1}^{m} p_i(x)\phi_i(t - x)u(t)dt = f(x),$$

where the $\phi_i(x)$ are such that the integrals $\int_x^{\infty} \phi_i(s)s^n ds$ exist for all values of n;

(b) $$u(x) + \int_a^b \sum_{i=1}^{m} p_i(x)q_i(t)u(x + ct)dt = f(x);$$

(c) $$p_m(x)u(x + m) + p_{m-1}(x)u(x + m - 1) + \cdots + p_0(x)u(x) = f(x);$$

(d) the Laplace differential equation of finite order.

These formal equivalences are put in evidence by means of expansions in Taylor's series.

* D. C. Lewis, Transactions of this Society, vol. 35 (1933), pp. 792–823.

† H. T. Davis, Annals of Mathematics, vol. 32 (1931), pp. 686–714.

4. *Equations with Constant Coefficients.* Here the first problem is that of solving the equation

$$a_0 y + a_1 y' + a_2 y'' + \cdots = \phi(x), \tag{15}$$

where the coefficients a_ν are constants, $\phi(x)$ is a given function and $y(x)$ is the function to be determined. If $y(x)$ is any solution, then the left member of equation (15) must be a convergent series; hence we must have

$$\limsup_{\nu=\infty} \left| a_\nu y^{(\nu)}(x) \right|^{1/\nu} \tag{16}$$

less than or equal to unity for every admissible value of x. When this superior limit is 1 there is uncertainty about convergence. Hence in a first study of (15) it seems natural to require that the superior limit (16) shall have a value less than 1; and we now impose this condition.

It seems natural also, in a first view of the problem, to attain this condition by hypotheses on the a_ν and the $y(x)$ taken separately, since the former are known constants and the latter is the unknown function. We may naturally bring this about by restricting the solutions $y(x)$, which are to be admitted, to functions $y(x)$ such that the superior limit

$$\limsup_{\nu=\infty} \left| y^{(\nu)}(x) \right|^{1/\nu} \tag{17}$$

shall be finite. Then we are led naturally to the following condition on the coefficients:

$$\limsup_{\nu=\infty} \left| a_\nu \right|^{1/\nu} = \sigma < \infty . \tag{18}$$

We therefore adopt this condition as one of the basic hypotheses.

The class of functions $y(x)$ and the basic hypothesis on the coefficients a_ν, to which we have thus been led in a natural way, have as a matter of fact played a central role in the theory of equations of the form (1).

If $y(x)$ is an analytic function, it is readily proved that the value of the superior limit (17) is independent of x. If this value is τ, we say that $y(x)$ is of exponential type τ.

The most elegant known theorem concerning solutions of equation (15) is perhaps the following.

THEOREM 1. *In the linear differential equation of infinite order,*

$$a_0y + a_1y' + a_2y'' + \cdots = \phi(x), \tag{15}$$

let the coefficients a_r be constants such that the function $F(z)$,

$$F(z) = a_0 + a_1z + a_2z^2 + \cdots, \tag{19}$$

is analytic in the region $|z| \leqq q$, where q is a given positive constant or zero, and let $\phi(x)$ be a function of exponential type not exceeding q. If $F(z)$ vanishes at least once in the region $|z| \leqq q$, let n be the number of its zeros in this region (each counted according to its multiplicity) and let $P(z)$ be the polynomial of degree n with leading coefficient unity such that $F(z)/P(z)$ does not vanish in the region. If $F(z)$ does not vanish in the region, let $P(z)$ be identically equal to 1. When $P(z) \equiv 1$, let $P_{n-1}(z)$ be identically equal to zero; otherwise let it be an arbitrary polynomial of degree $n-1$ (including the case of an arbitrary constant when $n=1$). Then the general solution $y(x)$ of (15), subject to the condition that it shall be a function of exponential type not exceeding q, may be written in the form

$$y(x) = \frac{1}{2\pi i}\int_{C_\rho} \frac{\psi(s)}{F(s)}\, e^{xs}ds + \frac{1}{2\pi i}\int_{C_\rho} \frac{P_{n-1}(s)}{P(s)}\, e^{xs}ds, \tag{20}$$

where

$$\psi(s) = \sum_{r=0}^{\infty} \frac{\phi^{(r)}(0)}{s^{r+1}}, \tag{21}$$

and where C_ρ is a circle of radius ρ about 0 as center, ρ being greater than q and such that $F(z)$ is analytic in the region $q > |z| \leqq \rho$ and does not vanish there.

If $\phi(x)$ is precisely of exponential type q, then the named solution $y(x)$ is also of exponential type q.

If we take $P_{n-1}(x) \equiv 0$, we have in (20) a particular solution of equation (15). On subtracting this particular solution from the general solution (20) we have the general solution (of exponential type not exceeding q) of the homogeneous equation obtained from (15) on replacing $\phi(x)$ by 0. If $F(z)$ does not vanish in the region $|z| \leqq q$, the latter solution is identically equal to zero. If $F(z)$ does vanish in this region, then the named solution of the homogeneous equation is identical with the gene-

ral solution of the differential equation $P(D)u=0$, where D denotes the derivative with respect to x.

It is desirable to give a proof of Theorem 1. The most elegant seems to be the following one, developed by G. B. Lang (in an unpublished Illinois dissertation) on the basis of methods which have been frequently employed by S. Pincherle over a period now stretching to almost fifty years. We present the demonstration merely in outline.

We write $\phi(x)$ and $y(x)$ in the forms

$$\phi(x) = \sum_{\nu=0}^{\infty} s_\nu \frac{x^\nu}{\nu!}, \qquad y(x) = \sum_{\nu=0}^{\infty} t_\nu \frac{x^\nu}{\nu!}.$$

Then we have the conditions

$$\limsup_{\nu=\infty} |s_\nu|^{1/\nu} \leqq q, \qquad \limsup_{\nu=\infty} |t_\nu|^{1/\nu} \leqq q. \tag{22}$$

Substituting in (15) and equating coefficients of like powers of x, we have the necessary conditions

$$\sum_{\mu=0}^{\infty} a_\mu t_{\mu+\nu} = s_\nu, \qquad (\nu = 0, 1, 2, \cdots). \tag{23}$$

With $\phi(x)$ and $y(x)$ we associate the functions

$$\psi(x) = \sum_{\nu=0}^{\infty} \frac{s_\nu}{x^{\nu+1}}, \qquad g(x) = \sum_{\nu=0}^{\infty} \frac{t_\nu}{x^{\nu+1}}.$$

When $g(x)$ is thus defined, subject only to the second relation in (22), then every function $y(x)$ of exponential type not exceeding q may be expressed in the form

$$y(x) = \frac{1}{2\pi i}\int_{C_\rho} g(s)e^{xs}ds, \tag{24}$$

as one may readily verify by aid of the expansion of e^{xs} in powers of s. Since $y(x)$ is to satisfy (15), we must now have conditions (23). By aid of these conditions it may be seen that the Laurent expansion of the function $F(x)g(x)-\psi(x)$ in powers of x contains no non-vanishing terms in negative powers of x.

This suggests that $y(x)$ in (24) be written in the form

$$y(x) = \frac{1}{2\pi i}\int_{C_\rho} \frac{\psi(s)}{F(s)} e^{xs}ds + \frac{1}{2\pi i}\int_{C_\rho} \frac{F(s)g(s) - \psi(s)}{F(s)} e^{xs}ds. \tag{25}$$

In the fraction in the second integrand multiply both numerator and denominator by $P(s)/F(s)$, a function which is analytic and does not vanish in the region $|s| \leqq \rho$. Then the last integrand takes the form $A(s)e^{xs}/P(s)$, where $A(s)$ is analytic and single-valued in the region $|s| \leqq \rho$. Therefore the second integral in (25) vanishes when $P(s) \equiv 1$; otherwise the sum of the principal parts of $A(s)/P(s)$ at its poles in the region $|s| \leqq q$ may be written in the form $Q(s)/P(s)$, where $Q(s)$ is a polynomial of degree $n-1$ at most. If $P(s) \equiv 1$, we still write $Q(s)/P(s)$, but understand that $Q(s) \equiv 0$ in this case. Then from (25) we see that the required solution necessarily has the form

$$y(x) = \frac{1}{2\pi i}\int_{C_\rho} \frac{\psi(s)}{F(s)}\, e^{xs} ds + \frac{1}{2\pi i}\int_{C_\rho} \frac{Q(s)}{P(s)}\, e^{xs} ds.$$

Since it is easy to verify directly that (20) affords a solution of (15), it follows from the result just obtained that (20) affords the general solution of (1) subject to the conditions which have been imposed.

Several investigators* have contributed to the development of the theory of linear differential equations of infinite order with constant coefficients and of systems of such equations. Several of the memoirs are devoted to the derivation of results very similar to those given in the foregoing theorem; and several methods have been given for their demonstration. Both the methods and the results for the case of equation (15) have been

* See, among others, the following: S. Pincherle, Memorie Reale Accademia delle Scienze, Bologna, (4), vol. 9 (1888), pp. 45–71; C. Bourlet, Annales de l'École Normale Supérieure, (3), vol. 14 (1897), pp. 133–190; S. Pincherle, Mathematische Annalen, vol. 49 (1897), pp. 325–382; T. Lalesco, Journal de Mathématiques, (6), vol. 4 (1908), pp. 125–202; J. F. Ritt, Transactions of this Society, vol. 18 (1917), pp. 21–26, 27–49; F. Schürer, Leipziger Berichte, vol. 70 (1919), pp. 185–246; H. von Koch, Arkiv för Matematik, Astronomi och Fysik, vol. 15, No. 26 (1921), pp. 1–16; S. Pincherle, Acta Mathematica, vol. 48 (1926), pp. 279–304; N. Wiener, Mathematische Annalen, vol. 95 (1926), pp. 557–584; G. Pólya, Göttinger Nachrichten, 1927, pp. 187–195; F. Lettenmeyer, Dissertation, München, 1927, pp. 1–52; I. M. Sheffer, Annals of Mathematics, vol. 30 (1929), pp. 250–264, and Transactions of this Society, vol. 31 (1929), pp. 261–280, 281–289; G. Valiron, Annales de l'École Normale Supérieure, (3), vol. 46 (1929), pp. 25–53; H. T. Davis, American Journal of Mathematics, vol. 52 (1930), pp. 97–108; M. Ghermanesco, Bulletin de l'Académie Royale de Belgique, vol. 19 (1933), pp. 387–394; G. B. Lang, Illinois dissertation, 1935 (unpublished).

extended to systems of such equations, or, more precisely, to systems of the form (4) in which the coefficients $a_{ik\nu}(x)$ are taken to be constants; and this has been carried out for systems having either a finite or an infinite number of equations. This development has, in the main, involved only what is a fairly natural extension of the results for a single equation. Some complications have arisen from the presence of exceptional cases, but it can hardly be said that they have been of a serious character. There has been no important change in the types of the functions appearing in the solutions.

If one notes the character of the functions which have appeared in these solutions, as indicated for instance by our Theorem 1, he may well be struck by the fact that no real novelties have appeared in passing from equations of finite order to equations of infinite order. In fact, in the case of homogeneous equations, the solutions which are set forth by means of the theorem are precisely the same as the solutions of certain related equations of finite order. By allowing the type q of the solution to increase one may indeed bring in more and more of these solutions without altering the given equation of infinite order, but at each stage he will still have only such functions as arise from equations of finite order. By means of the limiting processes which are thus suggested, however, one presumably would be able to introduce new classes of functions, but such functions are not explicit in Theorem 1. This character of the results obtained has persisted throughout a large part (but not all) of the development up to the present. Wherever this character of result persists, it seems fair to say that the theory so developed for differential equations of infinite order has not truly departed from the theory of equations of finite order, whence one would probably be led to suppose that a penetrating understanding of the extended field had not arisen. The pioneer in this more intensive study of differential equations of infinite order is J. F. Ritt.* We turn now to an account of the development due to Ritt and to those who have pursued similar questions further.

It is now convenient to write the equation in the form

$$(26) \qquad A(y) \equiv y + a_1 y' + a_2 y'' + \cdots + a_n y^{(n)} + \cdots = 0,$$

* Transactions of this Society, vol. 18 (1917), pp. 27–49.

whence the generating function $F(z)$ has the expansion

$$F(z) = 1 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots . \tag{27}$$

In the investigations now to be considered, the function $F(z)$ is an integral function which is subject to certain additional conditions. It is clear that if a is a zero of $F(z)$ of order μ, then $e^{az}Q(z)$, where $Q(z)$ is a polynomial of degree $\mu-1$, is a solution of (26). It will be called a fundamental solution.

In his remarkable memoir, Ritt has given (for the first time) general properties of the solution of (26) when the generating function is of genus zero. His results have been extended by G. Pólya* who has shown that, when $F(z)$ is of minimal type of order one, *the analytic solutions of* (26) *are holomorphic functions whose domains of existence are convex*. On the other hand, on making the hypothesis that $F(z)$ has only a finite number of multiple zeros and that the moduli of its zeros are sufficiently regular,† Ritt has shown that any solution whatever may be developed in a series of fundamental solutions valid in the whole domain of its existence.

By means of this result, Ritt gave for the first time the extension of the Fabry and Hadamard theorem on lacunary Taylor series to the case of series of the form

$$\sum c_n e^{\alpha_n z}.$$

If the constants $\alpha_1, \alpha_2, \cdots$ *satisfy the stated regularity condition, then the frontier of the domain of convergence of this series* (*a domain which is convex*) *is an essential cut for the function which it defines*. The demonstration of this theorem given by Ritt does not differ in any essential way from that which was independently given later by Landau and Carlson.

The later work of G. Valiron‡ on linear differential equations of infinite order with constant coefficients contains what seems to be the most important results yet developed along the lines

* Göttinger Nachrichten, 1927, pp. 187–195.

† If the zeros of $F(z)$ are denoted by $\alpha_1, \alpha_2, \alpha_3, \cdots$, then the regularity condition imposed by Ritt may be stated as follows. There exists an integer r such that for $n \geqq r$ we have $|\alpha_{n+1}/\alpha_n| > 1 + k/n$, where k is a suitable constant greater than 2. This condition is used by Ritt for the proof of certain needful properties of the generating function $F(z)$.

‡ Annales de l'École Normale Supérieure, (3), vol. 46 (1929), pp. 25–53.

of Ritt's path-breaking contribution. The work of Pólya had appeared in the meantime. Valiron set himself the problem of extending the results of Ritt. He removes the hypothesis that the number of multiple zeros of $F(z)$ is limited. Moreover his hypotheses on $F(z)$ are in other ways less restrictive than those of Ritt. He finds that all the results of Ritt remain valid if one supposes only that $F(z)$ is of minimal type of order one (a fact which Valiron communicated to Ritt before the appearance of Pólya's memoir). Some modifications of the results are necessary when $F(z)$ is of mean type of order one. The memoir of Valiron follows rather closely the methods of Ritt. It is rich in results other than those which we have explicitly indicated.

These investigations by Ritt and by Pólya and by Valiron mark the beginning of an intensive investigation of differential equations of infinite order along lines which bring to light essential features which are characteristic of the fact that the order of the equation is infinite. It seems clear that results of this character should be in the forefront of attention.

We shall close this section by stating without proof a hitherto unpublished general theorem from which a number of previously known results are readily obtained, but unfortunately none of those deep-lying theorems which Ritt and his followers have obtained. This theorem belongs to a range of ideas which have been especially emphasized by Pincherle.

Consider the equation (15). Let $F(z)$, as in (19), be the corresponding characteristic function. We suppose that $F(z)$ and $\phi(x)$ are both integral functions, and we write $\phi^{(\nu)}(0) = s_\nu$. The solution $y(x)$ of (15) is required to be an integral function. Let τ be a given positive number. Let $\{\lambda_\nu\}$ and $\{\mu_\nu\}$, $(\nu = 0, 1, 2, \cdots)$, be two infinite sequences of positive numbers such that $\tau \leqq \mu_\nu \leqq \lambda_\nu$ for every ν. For each particular value of ν let C_ν be a closed contour of finite length encircling the point 0 and lying in the ring $\mu_\nu \leqq |z| \leqq \lambda_\nu$, and let C_ν pass through no point at which $F(z)$ vanishes. Define T_ν by the relation

$$T_\nu = \sum_{k=0}^{\infty} |a_k| \lambda_\nu^k .$$

Then we have the following result.

A sufficient condition that $y(x)$, defined by the relation

$$y(x) = \sum_{\nu=0}^{\infty} \frac{s_\nu}{2\pi i} \int_{C_\nu} \frac{e^{zx} dt}{z^{r+1} F(z)},$$

shall be an integral function satisfying equation (15) *is that the series*

$$\sum_{\nu=0}^{\infty} e^{\rho\lambda_\nu} \mid s_\nu \mid T_\nu \mu_\nu^{-r-1} \int_{C_\nu} \frac{\mid dz \mid}{\mid F(z) \mid}$$

shall be convergent for every positive number ρ.

5. *Equations with Polynomial Coefficients.* The typical equation here is of the form

$$(28) \quad \sum_{n=0}^{\infty} (a_{n0} + a_{n1}x + a_{n2}x^2 + \cdots + a_{np}x^p) y^{(n)}(x) = \phi(x),$$

where p is a positive integer and not all the quantities a_{np} are zero. This is the special case of equation (1) in which the coefficients $a_k(x)$ are all polynomials of bounded degree. Corresponding systems (2) and (4) and the generalizations of them indicated in §2 also come into play here, the coefficients being restricted in each case to be polynomials of bounded degree. Such equations and systems of equations have formed the subjects of a considerable number of investigations.*

So far as I am aware, an intensive investigation of these equations along the line of what may be called the Ritt tradition has never been carried out. In fact I know of no effective pursuit of such questions other than those which belong to equations with constant coefficients. So far as the subject of the present section is concerned, the investigations up to date correspond

* See among others, the following: S. Pincherle, Memorie Accademia delle Scienze, Ist., Bologna, (4), vol. 9 (1888), pp. 181–204; C. Bourlet, Annales de l'École Normale Supérieure, (3), vol. 14 (1897), pp. 133–190; T. Lalesco, Journal de Mathématiques, (6), vol. 4 (1908), pp. 125–202; F. Schürer, Leipziger Berichte, vol. 70 (1919), pp. 185–240; E. Hilb, Mathematische Annalen, vol. 82 (1920), pp. 1–39; E. Hilb, Mathematische Annalen, vol. 84 (1921), pp. 16–30, 43–52; O. Perron, Mathematische Annalen, vol. 84 (1921), pp. 1–15, 31–42; H. von Koch, Arkiv för Matematik, Astronomi och Fysik, vol. 16 (1922), No. 6, 21 pp.; F. Lettenmeyer, Dissertation, München, 1927, pp. 1–52; I. M. Sheffer, Annals of Mathematics, vol. 30 (1929), pp. 345–372; I. M. Sheffer, Transactions of this Society, vol. 31 (1929), pp. 261–280; H. T. Davis, Annals of Mathematics, vol. 32 (1931), pp. 686–714.

in the main to aspects of the problem similar to those treated in our §4 exclusive of what has arisen from the use of the methods of Ritt. Even for these relatively simple questions concerning equation (28) no inconsiderable difficulties have been encountered.

Pincherle's detailed investigation, in the paper cited, was confined to a certain generalized difference equation with rational coefficients; but the methods employed by him, extending those which he had previously used for the case of constant coefficients, have played an important role in the subsequent development of the theory; and this is the reason for our calling attention to his work in the present connection. In several papers, not mentioned in our references, some other aspects of the theory of difference equations have been considered in such a way as to throw light on the theory of differential equations of infinite order.

It seems that the first explicit treatment of equation (28) is that contained in Lalesco's memoir of 1908, though results concerning it were implicit in certain earlier papers, particularly those of Bourlet and Pincherle both of whom had previously published several papers bearing indirectly on the problem. Lalesco, incidental to a consideration of the inversion of Volterra integrals, applied to the homogeneous equation the general Laplace transformation

$$y(x) = \int_L e^{xt} v(t) dt,$$

choosing the path of integration conveniently with reference to the coefficients in the equation. In 1920 and 1921 E. Hilb developed the method in more detail.

O. Perron (loc. cit., 1921) avoided the use of the Laplace transformation and the general theory of infinite matrices and gave a quite elementary development by means of the theory of systems of equations of the form

$$\sum_{n=0}^{\infty} (\alpha_n + \beta_{mn}) x_{m+n} = c_m, \qquad (m = 0, 1, 2, \cdots).$$

He assumed that $\phi(x)$ is of exponential type not exceeding q and that the $a_{\nu\lambda}$ are such that the functions

$$h_\lambda(z) = \sum_{\nu=0}^{\infty} a_{\nu\lambda} z^\nu, \qquad (\lambda = 0, 1, \cdots, p),$$

are regular for $|z| \leqq q$ (whence the convergence radii of these series are greater than q). Then he showed that the homogeneous equation, that for which $\phi(x) \equiv 0$ in (28), has exactly $\rho - p + s$ linearly independent solutions of exponential type not exceeding q, where ρ is the number of zeros of $h_p(z)$ in the circle $|z| \leqq q$ (multiple zeros counted multiply), and where s denotes the number of linearly independent solutions, regular in the region $|z| \leqq q$, of the auxiliary equation

$$\sum_{\lambda=0}^{p} h_\lambda(z) u^{(\lambda)}(z) = 0$$

of order p. He showed further that the differential equation (28) has, for every choice of the function $\phi(x)$ of exponential type not exceeding q, solutions y of exponential type not exceeding q when and only when the corresponding homogeneous equation has exactly $\rho - p$ integrals.

Lettenmeyer (loc. cit., 1927) extended the methods of Perron to systems of equations having polynomial coefficients of bounded degree.

E. Hilb (loc. cit., 1920, 1921) investigated equation (28) by aid of the infinite set of equations in an infinite number of unknowns obtained through unlimited differentiation of the equation. He obtained conditions assuring the uniqueness of a solution.

While his results were still unpublished Hilb communicated some of them to H. von Koch. These turned the latter's attention again to some ideas with which he had been concerned in 1912 and it turned out that they were capable of leading to elegant methods for dealing with equations (28) and also with the corresponding equations with constant coefficients. Thus, almost simultaneously, three different authors, under the influence of widely diverse guiding ideas, were investigating equations of the form (28). The work of F. Schürer in 1919 also bears on the same problem.

In 1929 I. M. Sheffer dealt with equations with constant coefficients and with linear coefficients. He also investigated expansions in generalized Appell polynomials and treated certain

related linear functional equations. The methods employed are similar to those which have been used by Pincherle for many years in the investigation of related problems. They arise essentially from the use of the Laplace transformation and the reduction of the equation to a contour integral equation by means of which the investigation may be carried forward. The method is extended to partial differential equations and to "Laurent differential equations," both with constant coefficients.

H. T. Davis (loc. cit., 1931) treated equation (28) by means of the calculus of operators, employing the symbolic methods used by Pincherle and especially by Bourlet. The formal solution of (28) thus obtained is reduced to three useful forms. There is a discussion of the validity of the formal solutions and the domain of functions to which the operators apply, the domain being extended beyond that considered by previous writers. "The difficulties admitted by this extension," the author adds, "have not been entirely resolved, however, since they have been discovered to be inherent in the nature of asymptotic and summable series, the theory of which is still obscure in many points."

On this note of inachieved results we take our leave of the important subject treated in this section, adding merely the statement of our judgment that the results attained up to the present have hardly penetrated the surface. The most characteristic elements of this theory, it would seem, still await elucidation in the future.

6. *Equations with Analytic Coefficients.* Much of the work of Pincherle and Bourlet, referred to in earlier sections, has important bearings on equations with analytic coefficients; we shall, however, omit further analysis of these contributions, confining our attention (in this section) to certain other papers which bear more specifically on the analytic function-theoretic aspects of the theory of linear differential equations of infinite order.

H. von Koch* considered a system of the form

$$\frac{dx_i}{dt} = \sum_{\nu=1}^{\infty} \alpha_{i\nu} x_\nu, \qquad (i = 1, 2, \cdots), \tag{29}$$

*Öfvers. Kongl. Svenska Vetenskaps-Akademien, Handlingar, (Stockholm), vol. 56 (1899), pp. 395–411.

where the $\alpha_{i\nu}$ denote power series in t which converge when $|t| < R$, and where $|\alpha_{i\nu}| < S_i A_\nu$ in the region $|t| \leqq \rho < R$, S_i and A_ν being independent of T and being such that the series $\sum S_k A_k$ converges. Let $x_1^{(0)}, x_2^{(0)}, \cdots$ be a sequence of constants such that the series $\sum x_k^{(0)} A_k$ converges absolutely. Then it is shown that there is one and only one integral system of power series $x_1, x_2, x_3, \cdots$ which for $t=0$ take respectively the initial values $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \cdots$. These series converge when $|t| < R$. Emphasis is placed on the fact that the notion of fundamental systems of solutions of (1) may be introduced and that a significant part of the Fuchs theory of linear differential equations may be carried over to these systems. Extensions of the theory are also presented.

Lalesco's paper of 1908, cited in §4, contains results relevant to the present section.

M. Gramegna* employed the matrix notation and certain symbolic processes in treating (29) and the corresponding non-homogeneous system.

F. R. Moulton† treated the infinite system of differential equations

$$\frac{dx_i}{dt} = f_i(t; x_1, x_2, \cdots) = a_i + f_i^{(1)} + f_i^{(2)} + \cdots, \qquad (i = 1, 2, \cdots), \tag{30}$$

where a_i is a constant and $f_i^{(j)}$ is the totality of terms in the power series expansion of f_i which are homogeneous in $t, x_1, x_2, \cdots$ and of degree j. The functions f_i are said to be of analytic type. It is assumed that the following hypotheses are satisfied:

(H_1) $x_i = 0$ at $t=0$ for each i of the set 1, 2, 3, $\cdots$;

(H_2) Finite real positive constants $c_0, c_1, c_2, \cdots, r_0, r_1, r_2, \cdots$. A, a exist such that

$$s = c_0 t + c_1 x_1 + c_2 x_2 + \cdots \tag{31}$$

converges if

* Atti Reale Accademia delle Scienze, Torino, vol. 45 (1910), pp. 469–491.

† Proceedings National Academy of Sciences, vol. 1 (1915), pp. 350–354. The substance of this investigation, together with some applications, is contained in the final chapter of Moulton's book on *Differential Equations*, 1930.

$$(32) \qquad |t| \leqq r_0, \qquad |x_i| \leqq r_i, \quad (i = 1, 2, \cdots),$$

and such that $Ar_i s^j$ dominates $f_i^{(j)}$ and $a_i \leqq Ar_i a$.

If an analytic solution of (30) exists, satisfying the initial conditions (H_1), it will necessarily have the form

$$x_i = A_i^{(1)} t + A_i^{(2)} t^2 + A_i^{(3)} t^3 + \cdots, \qquad (i = 1, 2, \cdots).$$

It is readily shown that just one formal solution of this form exists. In order to prove the suitable convergence of the series in this formal solution, and hence to establish the existence of an actual solution, the author employed the method of dominant functions in much the same way as that which is usual in the corresponding case of finite systems. The result obtained is a natural generalization of the classic theory of finite systems.

The limitations placed on t in carrying out the convergence proof are so restrictive that the corresponding x_i are not shown to attain the boundary of the region for which the right members of (30) converge. A question thus arises whether the solution can be continued beyond the domain indicated by the first argument. That the answer is affirmative is shown by the presentation of an effective method for obtaining the desired continuation.

W. G. Simon* proved general existence theorems for equations of the form (29) and in particular treated certain types of solutions of particular kinds of systems with periodic coefficients. He found that many of the phenomena of the finite systems are carried over into the infinite systems of differential equations.

W. Sternberg† treated the system

$$(33) \qquad \frac{dy_i}{dx} = a_{i0}(x) + \sum_{j=1}^{\infty} a_{ij}(x) y_j, \qquad (i = 1, 2, \cdots),$$

where the $a_{ik}(x)$ are analytic functions of x which are regular in the region S defined by the relation $|x-a| \leqq r$, where a is a given point and r is a given positive constant. It is assumed that the series in the relations

$$g_i(x) = |a_{i0}(x)| + |a_{i1}(x)| + |a_{i2}(x)| + \cdots, \quad (i = 1, 2, \cdots),$$

* American Journal of Mathematics, vol. 42 (1920), pp. 27–46.

† Heidelberg Akademie Sitzungsberichte, 1920, No. 10, pp. 1–21.

all converge uniformly in S. Then constants N_i exist such that we have $0 \leqq g_i(x) \leqq N_i$, $(i=1, 2, \cdots)$, throughout S. It is assumed that N exists such that $N_i \leqq N$, $(i=1, 2, \cdots)$. Let $b_1, b_2, \cdots$ be any set whatever of constants such that $|b_i| < C$, $(i=1, 2, \cdots)$, for some appropriate constant C. Then it is shown that there is one and only one system of integrals $y_1, y_2, \cdots$ of (33) which satisfy the initial conditions $y_i|^{x=0} = b_i$ and are regular throughout S. These integrals are bounded functions of x and throughout S satisfy for each i the inequality $|y_i(x)| < Ce^{Nx}$. The proof is made by the method of successive approximations.

The corresponding homogeneous system (namely, that for which $a_{i0}(x) \equiv 0$) is treated under the additional hypothesis that constants K and n, $n>1$, exist such that $|a_{ik}(x)| < K(ik)^{-n}$ in S, and it is shown that systems of solutions exist having the characteristic properties of fundamental systems. It is finally indicated that the essential results hold under somewhat less restrictive hypotheses.

It is clear that we have here (as in previous cases) a choice of hypotheses which lead to conclusions as nearly similar as possible to those subsisting in the finite case. What one would prefer to see are results which are characteristic of the infinite case. But these, in the main, await discovery in the future.

Infinite systems of differential equations have also been treated by A. Wintner,* D. C. Lewis,† and others. (See also the next section.)

Let us consider the (finite or infinite) differential operator $P(D)$ defined by the relation

$$P(D) = a_0D^0 + a_1D + a_2D^2 + a_3D^3 + \cdots, \tag{34}$$

where $a_0, a_1, a_2, \cdots$ are all given functions of x and where D is the symbol for differentiation with respect to x. The theory of linear differential equations of infinite order has brought to notice those operators $P(D)$ which have the following general property in common with the operation of differentiation, namely, that for any given finite point whatever x_0 and any

* Mathematische Annalen, vol. 95 (1926), pp. 544–556; vol. 98 (1928), pp. 273–280; American Journal of Mathematics, vol. 53 (1931), pp. 241–257.

† Transactions of this Society, vol. 35 (1933), pp. 792–823.

given function $u(x)$ whatever, analytic at x_0, the operator $P(D)$ shall be applicable to $u(x)$ and shall yield a resulting function

$$P(D)u \equiv a_0u + a_1u' + a_2u'' + \cdots$$

which is itself analytic at x_0. By considering the functions $u(x) = (x-x_0)^k$, for varying points x_0 and varying non-negative integers k, it is easy to show that each of the functions $a_0, a_1, a_2, \cdots$ is an integral function.

In fact it is not difficult to show that for the required conditions on $P(D)$ it is necessary that the functions $a_0, a_1, a_2, \cdots$ shall be integral functions which for every x_0 verify the relation

$$\lim_{\nu=\infty} \left| a_\nu(x_0)\nu! \right|^{1/\nu} = 0.$$

On the other hand, it may be shown to be sufficient that the functions $a_0, a_1, a_2, \cdots$ shall be integral functions which for every non-negative number σ verify the relation

$$\lim_{\nu=\infty} (M_{\sigma\nu}\nu!)^{1/\nu} = 0,$$

where $M_{\sigma\nu}$ is the maximum value of $|a_\nu(x)|$ for $|x| = \sigma$.

Whether the necessary condition and the sufficient condition here given can be brought closer together by simple means I have not determined, even though the results as they stand are not altogether satisfactory. The reader will not find it difficult to supply the proof for at least as much as is stated here.

7. *Some Further Problems and Connections.* The problem of an infinite system of differential equations was treated by E. H. Moore* at the fourth international congress of mathematicians at Rome in 1908, from the point of view of general analysis, the functions not being restricted to those of analytic type. In a series of papers W. L. Hart† developed theorems concerning a type of real-valued functions of infinitely many real variables and employs these results in treating infinite systems of ordinary differential equations both non-linear and linear; the linear

* Atti del IV Congresso Internazionale Matematica, vol. II (1909), pp. 98–114.

† Proceedings National Academy of Sciences, vol. 2 (1916), pp. 309–313; Transactions of this Society, vol. 18 (1917), pp. 125–160; American Journal of Mathematics, vol. 39 (1917), pp. 407–424.

system is associated with its adjoint, and the notion of fundamental systems of solutions is employed. T. H. Hildebrandt* developed a theory of linear differential equations in general analysis, utilizing both the concepts of general analysis and the general theory of integral equations. P. Flamant† has applied the method of successive approximations to equations of the form (33), pointing out that the results are valid both for the case of analytic functions and for that of continuous functions of a real variable.

W. L. Hart‡ developed the Cauchy-Lipschitz method for infinite systems of differential equations. I. A. Barnett§ treated both ordinary and partial differential equations with a continuous infinitude of variables. H. T. Davis‖ has associated fractional operations with problems of the type here treated. L. Pomey¶ has published a series of relevant papers. N. Wiener** has treated the operational calculus. W. T. Reid†† has developed a theory of infinite systems with auxiliary boundary conditions; he has also treated infinite systems in the domain of Lebesgue summable functions. Finally, there are many other papers dealing with our subject; they are written both by the authors here quoted and by others; considerations of space prevent our treating them; for the most part they will come to the reader's attention through references given in the papers referred to in this address.

There are several aspects of the theory of difference equations of finite order which illuminate and are illuminated by the theory of differential equations of infinite order, as will be seen from the relevant papers of Pincherle, Hilb, Perron, Carmichael,

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* Transactions of this Society, vol. 18 (1917), pp. 73–96, and vol. 19 (1918), pp. 97–108.

† Bulletin des Sciences Mathématiques, (2), vol. 45 (1921), pp. 81–87.

‡ American Journal of Mathematics, vol. 43 (1921), pp. 226–231.

§ American Journal of Mathematics, vol. 44 (1922), pp. 172–190, and vol. 45 (1923), pp. 42–53.

‖ American Journal of Mathematics, vol. 46 (1924), pp. 95–109, and vol. 49 (1927), pp. 123–142.

¶ Paris Comptes Rendus 1926, 1927, 1928, and elsewhere.

** Mathematische Annalen, vol. 95 (1926), pp. 557–584.

†† Transactions of this Society, vol. 32 (1930), pp. 284–318; Annals of Mathematics, vol. 32 (1931), pp. 37–46.

Bochner, Ghermanesco, and others; but we cannot here treat them further than the bare indication given in §3. It may be mentioned, however, that some of the theorems about differential equations of infinite order imply corresponding results about difference equations of infinite order, but ordinarily they do not yield the latter results under the most natural hypotheses. The suggestion is inevitable that we are in need of a more direct theory of difference equations of infinite order. A few results looking in this direction appear in the dissertation of G. B. Lang (already mentioned). Integral equations and integro-differential equations have been associated with differential equations of infinite order. In this connection one may mention an important paper by F. Schürer* on a common method of treating certain problems involving functional equations.

Problems relating to expansions in Appell polynomials are intimately connected with the theory of linear differential equations of infinite order with constant coefficients, a fact which was clearly recognized by Pincherle about fifty years ago. Recently, I. M. Sheffer† has developed a theory of expansions in generalized Appell polynomials‡ and has applied the results to a class of related linear functional equations. He was primarily concerned with linear differential equations of infinite order with polynomial coefficients of bounded degree and their relation to expansions in generalized Appell polynomials. Let

$$A_i(t) \sim \sum_{n=0}^{\infty} \alpha_{in} t^n, \quad (i = 0, 1, 2, \cdots, k),$$

be $k+1$ formal power series, with $A_k(t) \not\equiv 0$. Then the generalized Appell polynomials $\{G_n(x)\}$, of order k, are defined by the (formal) expansion

$$e^{tx}\{A_0(t) + xA_1(t) + \cdots + x^k A_k(t)\} \sim \sum_{n=0}^{\infty} G_n(x) t^n.$$

A general theory is given for the expansion of functions $f(x)$ of exponential type in terms of the polynomials $G_n(x)$, this theory being developed in intimate connection with differential equa-

* Leipziger Berichte, vol. 70 (1919), pp. 185–240.

† Transactions of this Society, vol. 31 (1929), pp. 261–280.

‡ Pincherle also considered certain generalized Appell polynomials.

tions of the type mentioned earlier in this paragraph. By playing the two theories one against the other the author makes useful extensions of both of them. Related expansion problems are likely to undergo a considerable development in the near future.

We may fittingly bring these remarks to a close by emphasizing one aspect of the existing theory of differential equations of infinite order to which we have already directed attention. In the main it has been true that the results mentioned are closely analogous to corresponding ones for differential equations of finite order. The theorems for the most part take the form which is suggested by corresponding theorems for the simpler case. In fact, there often seems to be a conscious purpose to formulate the hypotheses in such a way as to preserve the analogies in the most intimate form possible. This is perhaps to be expected in a first approach to the more general subject; it is natural to inquire to what extent analogies subsist. Again, in so far as explicit information about the functions in a solution is concerned, we usually have only that which arises in close analogy with the finite case. In fact, in all our remarks (except for differences shown by examples) it has been true, with one single exception, that the results presented indicate no marked departure from the finite case. So far as I am aware, there is no other general exception to this statement to be found in the literature. The exception to which we refer is in the results initiated and inspired by J. F. Ritt, as already indicated in our §4. In these results of Ritt and Pólya and Valiron we have properties of the solutions of linear differential equations of infinite order which are different from anything that arises in the finite case. They are characteristic of the equation with respect to its being of infinite order. It would seem to be beyond dispute that such characteristic properties are the ones most eagerly to be sought. And yet a single small group of them stands out as apparently unique in the whole literature. This fact presents a challenge to analysts interested in functional equations. We have hardly penetrated the surface of the important theory of differential equations of infinite order. We need a further development of those elements of the theory of these equations which are characteristic of them as being of infinite order.

THE UNIVERSITY OF ILLINOIS

A NECESSARY CONDITION FOR APPROXIMATION BY RATIONAL FUNCTIONS

BY J. L. WALSH

It is the object of the present note to establish the following two theorems; terminology is uniform with that of the writer's recent book on approximation:*

THEOREM 1. *In the extended z plane let R be a region whose boundary is denoted by B. Let every component of B either separate the plane into at least two regions or contain in each of its neighborhoods points of an infinite number of components of B each of which separates the plane into at least two regions. Let the function $f(z)$ be single-valued and analytic in R in the neighborhood of B, and let $\lim_{z_k \to z_0} f(z_k)$ exist and be equal to zero whenever the points z_k lie interior to R and approach a point z_0 of B. Then the function $f(z)$ vanishes identically interior to R in the neighborhood of B.*

THEOREM 2. *Let C be an arbitrary closed point set of the extended plane, and let points z_k (not necessarily denumerable) be given exterior to C. A necessary and sufficient condition that a function $f(z)$ single-valued and analytic on C can be uniformly approximated as closely as desired on C by a rational function whose poles lie in the points z_k is that $f(z)$ can be extended analytically from C so as to be single-valued and analytic in every point of the plane which is separated by C from the points z_k. That is to say, the condition is that there should exist a function which is single-valued and analytic not merely on C but also in every point of the plane separated by C from the points z_k, and which coincides with $f(z)$ on C.*

These theorems are slightly more general than the corresponding theorems that are given in the book just mentioned (loc. cit., §1.9, Theorem 15; §1.10, Theorem 16). The present Theorem 2 seems to be the definitive result in its field.

The sufficiency of the condition of Theorem 2 has already

* *Interpolation and Approximation by Rational Functions in the Complex Domain*, Colloquium Publications of this Society, vol. 20, 1935.

been established (loc. cit., §1.6, Theorem 8). The necessity of that condition will be proved by use of Theorem 1, and Theorem 1 is to be proved by application of quite recent results on harmonic measure due to R. Nevanlinna.*

I am indebted to my colleague Professor L. Ahlfors for this proof of Theorem 1. I formulated Theorem 1 as something more than a conjecture to Professor Ahlfors; he at once suggested the proof now to be given.

Assume B finite, which involves no loss of generality. The neighborhood of B in Theorem 1 may be chosen (loc. cit., §1.3, Theorem 4, and method of §1.5, Theorem 7) as a finite number of mutually exclusive subregions $S_1, S_2, \cdots, S_n$ of R, each region S_k bounded interior to R by a single Jordan curve, and bounded otherwise by one or more components of B. Every point z interior to R and within a certain distance $\delta > 0$ of B lies interior to some S_k.

The capacity of a point set is monotonically non-decreasing with the point set. The boundary of any limited region has positive capacity. Each component of B either separates the plane into at least two regions or contains limit points of components of B so separating the plane. Hence the totality of components of B which bound each S_k have positive capacity. The function $f(z)$ is bounded in S_k, if the Jordan curves bounding all the regions S_j are suitably chosen. Consequently (Nevanlinna, loc. cit.; $f(z)$ is bounded in S_k, hence *beschränktartig*; its boundary values on B vanish, thus form a set of harmonic measure zero) the function $f(z)$ vanishes identically in each S_k, therefore vanishes identically throughout the given neighborhood of B.

Theorem 1 is now completely proved and will serve in the proof of the necessity of the condition of Theorem 2.

Let $r_n(z)$ be a sequence of rational functions whose poles lie in the prescribed points z_k and which converges uniformly to $f(z)$ on C. Let R' be any one of the regions into which C separates the plane which is also separated by C from the points z_k, and let B' be the boundary of R'. Let S be the point set consisting of the components of B' each of which effectively

* Proceedings of the Eighth Scandinavian Congress (Stockholm, 1934) of Mathematicians (Lund, 1935), pp. 116–133, especially pp. 129–130.

separates R' from points z_k, and let $\overline{S}$ be the set composed of S plus its limit points. Every point of $\overline{S}$ is a point of B', hence a point of C. Let R be the region bounded by $\overline{S}$, such that every point of R' is a point of R. Every component of $\overline{S}$ either separates the plane into at least two regions or contains in each of its neighborhoods points of an infinite number of components of $\overline{S}$ each of which separates the plane into at least two regions.

The sequence $r_n(z)$ converges uniformly to $f(z)$ on C and on $\overline{S}$. Each function $r_n(z)$ is analytic in the closed region $\overline{R}=R+\overline{S}$, so the sequence $r_n(z)$ converges uniformly in the closed region $\overline{R}$ to some function $F(z)$, which is analytic in R and continuous in $\overline{R}$, and coincides with $f(z)$ on $\overline{S}$. The function $f(z)$ is analytic on C, thus (loc. cit., §1.5, Theorem 7) can be extended analytically from C into R in the neighborhood of $\overline{S}$. The function $F(z)-f(z)$ is analytic in the neighborhood of $\overline{S}$ and approaches zero whenever z interior to R approaches $\overline{S}$. It follows from Theorem 1 that $F(z)-f(z)$ vanishes identically in the neighborhood of $\overline{S}$, so the analytic extension of $f(z)$ from $\overline{S}$ or from C into the interior of R coincides with $F(z)$. This is true for every region R, so the proof is complete. Indeed, we have shown that $f(z)$ can be extended from C so as to be single-valued and analytic not merely throughout every region R' but throughout every region R.

From Theorem 2 follows without difficulty (loc. cit., §1.10, Theorem 17) the following theorem.

THEOREM 3. *Let C be an arbitrary closed point set of the extended plane, and let points z_k (not necessarily denumerable) be given not belonging to C. A necessary and sufficient condition that every function $f(z)$ analytic on C can be uniformly approximated as closely as desired on C by a rational function whose poles lie in the points z_k is that at least one point z_k lie in each of the regions into which C separates the plane.*

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ASSOCIATED ALGEBRAIC AND PARTIAL DIFFERENTIAL EQUATIONS

BY J. A. GREENWOOD

1. *Introduction.* Between an algebraic equation and an ordinary linear homogeneous differential equation with constant coefficients, in a single unknown, there is a familiar and useful association, illustrated by the equations

$$x^2 + 3x - 4 = 0, \qquad u'' + 3u' - 4u = 0.$$

Riquier* has shown that a similar association exists between finite systems consisting of m algebraic equations in n unknowns and m linear homogeneous partial differential equations in a single unknown and n independent variables as follows. Let

$$(1) \qquad \sum_0^p a^{\alpha}_{i_1 \cdots i_n}(i_1 \cdots i_n) = 0, \qquad (\alpha = 1, \cdots, m),$$

be any system of algebraic equations, where the a's are constants and $(i_1 \cdots i_n)$ is to be interpreted as the monomial $x_1^{i_1} \cdots x_n^{i_n}$. Let the associated system of partial differential equations be

$$(2) \qquad \sum_0^p a^{\alpha}_{i_1 \cdots i_n}(i_1 \cdots i_n)u = 0, \qquad (\alpha = 1, \cdots, m),$$

where $(i_1 \cdots i_n)$ is to be interpreted as the differential operator $\partial^{i_1+\cdots+i_n}/\partial x_1^{i_1} \cdots \partial x_n^{i_n}$.

We shall prove in this paper the two following theorems.

THEOREM 1. *System* (1) *is inconsistent if and only if the general solution of* (2) *is* $u = 0$.

THEOREM 2. *The general solution of* (2) *is a non-zero polynomial if and only if* $x_1 = \cdots = x_n = 0$ *is the solution of* (1).

2. *Corresponding Operations on the Two Systems.* We shall mul-

* *Sur la résolution numérique du système d'équations algébriques entières à un nombre quelconque d'inconnues*, Annales Scientifiques de l'École Normale Supérieure, vol. 63 (1928), pp. 145–188.

tiply the equations by $(j_1 \cdots j_n)$ and then form linear combinations of them with constant coefficients. The interpretation of the $(j_1 \cdots j_n)$ is to be that of the system into which it is multiplied. This gives rise to new associated systems of equations.

3. *System* (1) *Inconsistent.* Let us assume (1) is inconsistent. Then according to a known theorem* there exist for system (1) C's satisfying the relation

$$1 \equiv \sum_{\alpha}^{1,\cdots,m} C^{\alpha}\left(\sum_{0}^{p} a^{\alpha}_{i_1 \cdots i_n}(i_1 \cdots i_n)\right), \tag{3}$$

where

$$C^{\alpha} = \sum_{0}^{r} k^{\alpha}_{i_1 \cdots i_n}(i_1 \cdots i_n). \tag{4}$$

The corresponding result on (2) would of course be obtained by multiplying both sides of the equations (3) on the right by u. Then (2) implies $u = 0$ and from the form of (2) it is seen to be an actual solution.

If (1) has a solution $(\bar{x}_1, \cdots, \bar{x}_n)$, then (2) has the obvious *immediate*† solution $c \exp(\bar{x}_1 x_1 + \cdots + \bar{x}_n x_n)$, where c is an arbitrary constant. On the assumption that $u = 0$ is the general solution of (2), it follows immediately from this that (1) must be inconsistent. Thus Theorem 1 is established.

4. *System* (1) *Consistent with only the Trivial Solution.* Now consider the case where $x_1 = \cdots = x_n = 0$ is the solution of (1). According to Hilbert's zero theorem there exist C's similar to (4) and positive integers ρ_j such that

$$x_j^{\rho_j} \equiv \sum_{\alpha} C^{\alpha_j}\left(\sum_{0}^{p} a^{\alpha}_{i_1 \cdots i_n}(i_1 \cdots i_n)\right), \tag{5}$$

$$(\alpha = 1, \cdots, m; j = 1, \cdots, n).$$

Again, multiplying both sides of (5) on the right by u and interpreting $x_j^{\rho_j}$ as in (2), we see that the left member becomes $\partial^{\rho_j} u / \partial x_j^{\rho_j}$. System (2) implies that these derivatives vanish

* B. L. Van der Waerden, *Moderne Algebra*, vol. 1, 1930, p. 10.

† Riquier, loc. cit.

identically. Observe that there must be a non-zero solution of (2) else we have a contradiction with Theorem 1.

Consider any derivative $(i_1 \cdots i_n)$ whose order equals or exceeds $\rho_1 + \cdots + \rho_n$, that is,

$$(i_1 - \rho_1) + \cdots + (i_n - \rho_n) \geqq 0.$$

Then $i_j \geqq \rho_j$ for at least one j. Hence every derivative of the order considered is zero. Since there are but a finite number of derivatives below a given order taken with respect to a finite number of variables, it is clear that the series expansion solution for u reduces to a non-zero polynomial. If we assume that a non-zero polynomial is the general solution of (2), then (1) cannot have more than the trivial solution. For if it did, (2) would have a non-constant exponential solution. But if (1) has no solution our present hypothesis contradicts Theorem 1. Therefore (1) has the trivial solution and that alone, and Theorem 2 is proved.

We give an example illustrating the second result. Let (1) be

$$x_1^3 + x_2^3 = 0, \qquad x_1^3 - x_2^3 = 0,$$

and its associated system of partial differential equations (2) be

$$\frac{\partial^3 u}{\partial x_1^3} + \frac{\partial^3 u}{\partial x_2^3} = 0, \qquad \frac{\partial^3 u}{\partial x_1^3} - \frac{\partial^3 u}{\partial x_2^3} = 0.$$

Clearly $\rho_1 = \rho_2 = 3$. Thus the series expansion for u reduces to

$$u = \sum_0^2 a_{i_1 i_2} x_1^{i_1} x_2^{i_2},$$

where the coefficients are arbitrary constants.

An interesting application of the results obtained in this paper is that the conditions given by Riquier* for the existence of a solution of a system of algebraic equations (1) can now be extended to read: *The system* (1) *is consistent if and only if there exists a non-zero solution of* (2). In other words, *if* (2) *has any non-zero solution, it also has a non-zero immediate solution.*

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* Riquier, loc. cit.

A THEOREM IN THE THEORY OF SUMMABILITY

BY J. D. HILL

1. *Introduction.* Knopp* has developed a general integral form for linear methods of summability which includes as special cases all such methods which to date have found any considerable application. He establishes sufficient conditions for the regularity of his method but does not completely treat the question of their necessity. It may be of interest to observe that if Knopp's integral is interpreted in the sense of Lebesgue one may readily construct an example of his method which is regular in the class of bounded, measurable functions, but which does not satisfy condition (b)† of his regularity theorem. In conformity with Knopp's notation, let the curves $\mathfrak{C}_x$, $\mathfrak{C}_y$ be taken as the real axes $0 \leqq x < \infty$, $0 \leqq y < \infty$, respectively. Denote by $(\mathfrak{A})$ the class of all complex functions $f(x) \equiv f_1(x) + if_2(x)$ defined on $\mathfrak{C}_x$ such that $f_1(x)$, $f_2(x)$ are bounded and measurable on the interval $0 \leqq x \leqq X$ for every $X > 0$, and such that $\lim_{x\to\infty} f(x) \equiv L_f$ exists. Finally, let the function $K(x, y)$ be defined as $(-1)^j$ for $n-1 \leqq y < n$, $(n = 1, 2, 3, \cdots)$, $(j-1)/2^n \leqq x < j/2^n$, $(j = 1, 2, \cdots, 2^n)$; as $1/(y+1)$ for $1 \leqq x \leqq y+1$, $0 \leqq y < \infty$; and as zero for $0 \leqq y < x-1$, $1 < x < \infty$. Then for every $f(x) \subset (\mathfrak{A})$, $g(y) \equiv \int_0^{y+1} K(x, y) f(x)dx$ clearly exists on $\mathfrak{C}_y$, and we have

$$g(y) = \int_0^1 K(x, y) f(x)dx + \int_1^{y+1} \frac{1}{y+1} f(x)dx,$$

where the second integral tends to L_f as $y \to \infty$, since the function $1/(y+1)$ satisfies the conditions of Knopp's regularity theorem. Moreover, given $\epsilon > 0$, there exist step-functions $s_j(x)$ such that $\int_0^1 |f_j(x) - s_j(x)| dx < \epsilon$, $(j = 1, 2)$. Consequently, in view of $|K(x, y)| \leqq 1$, we have

* Knopp, *Zur Theorie der Limitierungsverfahren*, Mathematische Zeitschrift, vol. 31 (1929–30), pp. 97–127. To save space we assume that the reader is familiar with this paper.

† Loc. cit., p. 101.

$$\left|\int_0^1 K(x, y)f(x)dx\right| \leqq \left|\int_0^1 K(x, y)s_1(x)dx\right| + \left|\int_0^1 K(x, y)s_2(x)dx\right| + 2\epsilon;$$

$s_1(x)$, $s_2(x)$ being step-functions, the two integrals on the right tend to zero as $y\to\infty$. Hence $\lim_{y\to\infty} g(y) = L_f$ and $K(x, y)$ defines a regular method in the class $(\mathfrak{A})$. On the other hand, $\int_0^1 |K(x, y)|\,dx$ does not tend to zero as $y\to\infty$, so that condition (b) in this case is not fulfilled.

This fact leads us to consider a slight modification of Knopp's method, based on the Lebesgue integral, for which it is possible to state conditions necessary as well as sufficient for regularity in the class of functions defined below. Aside from specifying the type of integral used, the modification consists mainly in dispensing with the curves $\mathfrak{C}_x$, $\mathfrak{C}_y$ and directing attention to the real and imaginary parts of the complex functions with which we are concerned.

2. *K-Method of Summability.* Let $(\mathfrak{B})$ denote the class of all complex functions $f(x) \equiv f_1(x) + if_2(x)$ defined on the interval $I \equiv (0 \leqq x < \xi)$, $\xi \leqq \infty$, such that (i) $f_1(x)$, $f_2(x)$ are measurable on the sub-interval $0 \leqq x \leqq X$ for every X, $0 < X < \xi$, (ii) $|f(x)|$ is essentially bounded on I, and (iii) $f(x)$ tends to a limit L_f as $x\to\xi$ on I. Let $K(x, y) \equiv K_1(x, y) + iK_2(x, y)$ be an arbitrary complex function defined on the intervals I and $J \equiv (0 \leqq y < \eta)$, $\eta \leqq \infty$. We shall say that $K(x, y)$ defines a regular method of summability in the class $(\mathfrak{B})$ if for every $f(x) \subset (\mathfrak{B})$ the integral

$$\begin{aligned} U_y(f) &\equiv \int_0^\xi K(x, y)f(x)dx \\ &\equiv \lim_{X\to\xi} \int_0^X [K_1(x, y)f_1(x) - K_2(x, y)f_2(x)]dx \\ &\quad + i\lim_{X\to\xi} \int_0^X [K_1(x, y)f_2(x) + K_2(x, y)f_1(x)]dx \end{aligned}$$

exists on J, and if $U_y(f)$ tends to a limit equal to L_f as $y\to\eta$ on J. For brevity, the process of summability so defined will be called a K-method.

THEOREM. *For the regularity of a K-method in the class* $(\mathfrak{B})$ *it is necessary and sufficient that* (a) $\int_0^\xi K(x, y)dx$ *exist on* J *and tend to* 1 *as* $y \to \eta$ *on* J; (b) $\int_H K(x, y)dx \to 0$ *as* $y \to \eta$ *on* J, *for every measurable set* H *on* I *not having* ξ *as a limit point*; (c) $\int_0^\xi |K(x, y)|dx$ *exist on* J, *and that constants* M *and* Y *exist such that this integral is bounded by* M *for* $Y < y < \eta$.

PROOF. *Necessity.* Since the function $g(x) \equiv 1$ on I belongs to $\mathfrak{B}$ with $L_g = 1$, the necessity of (a) is apparent. Moreover, if $h(x)$ is the characteristic function of any set H of the type described in (b), then $h(x) \subset (\mathfrak{B})$ with $L_h = 0$; thus (b) is necessary. To establish (c) we observe first that the existence of $\int_0^\xi K(x, y)dx$ implies that $\int_0^X |K_1(x, y)|dx$, $\int_0^X |K_2(x, y)|dx$ both exist, from which the existence of $\int_0^X |K(x, y)|dx$ follows for y on J and $0 < X < \xi$. Suppose now, contrary to the first part of (c), that for some value of y, say y_0, we have $\int_0^\xi |K(x, y_0)|dx \equiv \lim_{X \to \xi} \int_0^X |K(x, y_0)|dx = \infty$. Then there exists on I a sequence $\{x_j\}$ tending to ξ, such that

$$\int_{x_{j-1}}^{x_j} |K(x, y_0)|dx > j, \qquad (j = 1, 2, 3, \cdots; x_0 = 0).$$

If we define $k(x)$ for $x_{j-1} \leqq x < x_j$, $(j = 1, 2, 3, \cdots)$, as zero if $K(x, y_0) = 0$, and as $[K_1(x, y_0) - iK_2(x, y_0)]/(j|K(x, y_0)|)$ if $K(x, y_0) \neq 0$, then $k(x) \subset (\mathfrak{B})$, whereas, for $n = 1, 2, 3, \cdots$,

$$\int_0^{x_n} K(x, y_0)k(x)dx = \sum_{j=1}^{n} \frac{1}{j} \int_{x_{j-1}}^{x_j} |K(x, y_0)|dx > n,$$

so that $U_{y_0}(k)$ does not exist. This contradiction proves the first part of (c). To establish the second part we first define $\text{ess.sup.}_{0 \leqq x < \xi} |f(x)|$ as the greatest lower bound of all numbers C for which $|f(x)| \leqq C$ almost everywhere. Then by introducing the functional $\|f\| \equiv \text{ess.sup.}_{0 \leqq x < \xi} |f(x)|$ as the norm, one may verify that $(\mathfrak{B})$ becomes a Banach space and that $U_y(f)$, for each fixed y on J, is a linear operation in this space. Now choose X arbitrarily, $0 < X < \xi$, and define $f_y(x)$ for $0 \leqq x \leqq X$ as 1 if $K(x, y) = 0$, as $[K_1(x, y) - iK_2(x, y)]/|K(x, y)|$ if $K(x, y) \neq 0$, and for $X < x < \xi$ as zero. For each value of y we have $f_y(x) \subset (\mathfrak{B})$ with $\|f_y\| = 1$. Hence we obtain $U_y(f_y) = \int_0^X |K(x, y)|dx \leqq \|U_y\| \cdot \|f_y\| = \|U_y\|$, from which it follows that

$$\int_0^{\xi} |K(x, y)|\, dx \leqq \|U_y\| \text{ for } y \text{ on } J, \tag{1}$$

since X was arbitrary. Let $s \equiv \{y_n\}$ be any sequence on J tending to η. By the assumed regularity, the sequence $\{U_{y_n}(f)\}$ converges for every $f \subset (\mathfrak{B})$, and hence by a theorem of Banach* there exists a number N, depending on s, such that $\|U_{y_n}\| \leqq N$ for $n = 1, 2, 3, \cdots$. From this fact we infer the existence of numbers M and Y such that $\|U_y\| \leqq M$ for $Y < y < \eta$, which, by virtue of (1), completes the proof of necessity.

Sufficiency. Assuming that $K(x, y)$ satisfies (a), (b), and (c), it is clear first of all that $U_y(f)$ exists on J for every $f \subset (\mathfrak{B})$. Now choose $f(x) \subset (\mathfrak{B})$ arbitrarily and set $f(x) \equiv L_f + r(x)$, where, accordingly, $r(x) \subset (\mathfrak{B})$ with $L_r = 0$. Then $U_y(f) = U_y(L_f) + U_y(r)$ and since by condition (a), $U_y(L_f) \to L_f$ as $y \to \eta$ on J, it remains only to show that $U_y(r) \to 0$. To this end let $\epsilon > 0$ be given; then a number $x_0(\epsilon)$ exists, $0 < x_0 < \xi$, such that

$$|r(x)| \leqq (2)^{1/2}\epsilon/(12M) \text{ for } x_0 < x < \xi. \tag{2}$$

Furthermore, if $r(x) \equiv r_1(x) + ir_2(x)$, there exist functions $R_j(x)$ of the form $R_j(x) \equiv \sum_{n=1}^{m} a_{nj} h_n(x)$, where the a_{nj} are real numbers and the $h_n(x)$ are the characteristic functions of certain measurable sets H_n on the interval $(0, x_0)$, such that

$$|r_j(x) - R_j(x)| \leqq (2)^{1/2}\epsilon/(12M) \text{ almost everywhere on } (0, x_0), \tag{3}$$

when $j = 1$ or 2. Setting $R(x) \equiv R_1(x) + iR_2(x)$, we may write

$$U_y(r) = \sum_{n=1}^{m} (a_{n1} + ia_{n2}) \int_{H_n} K(x, y)dx + \int_0^{x_0} K(x, y)[r(x) - R(x)]dx + \int_{x_0}^{\xi} K(x, y)r(x)dx.$$

By condition (b) a number $Y_0(\epsilon)$ exists, $Y \leqq Y_0 < \eta$, such that the first term on the right does not exceed $\epsilon/3$ in absolute value for $Y_0 < y < \eta$. Finally, making use of (2), (3), and condition (c), one readily shows that the remaining integrals are each numerically $\leqq \epsilon/3$ for $Y_0 < y < \eta$. Hence $|U_y(r)| \leqq \epsilon$ for $Y_0 < y < \eta$, which completes the proof of the theorem.

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* Banach, *Théorie des Opérations Linéaires*, p. 80, Théorème 5.

TRANSFORMS OF FUCHSIAN GROUPS*

BY P. K. REES

This paper gives four theorems concerning the relative sizes of the isometric circles of the transformations, $T(z) = (az+\bar{c})/(cz+\bar{a})$, of a Fuchsian group and those of the transforms, $S(z) = GTG^{-1}(z) = (Az+\overline{C})/(Cz+\overline{A})$, of T in which $G(z) = (\alpha z+\bar{\nu})/(\nu z+\bar{\alpha})$ is considered as fixed and T any transformation of the Fuchsian group.

THEOREM 1. *The necessary and sufficient condition that the radii, r_s and r_t, of the isometric circles of S and T be equal is that the midpoint, $(a-\bar{a})/(2c) = m$, of the line segment joining the centers, g_t and g_t', of the isometric circles, I_t and I_t', of T and T^{-1} be on the circle $Q_5(z)$ with the origin and the center, $g = -\bar{\alpha}/\nu$, of the isometric circle of G as opposite ends of a diameter or on the circle $Q_6(z)$ with the origin and $1/\bar{g}$ as opposite ends of a diameter.*

PROOF. The equations of $Q_5(z)$ and $Q_6(z)$ are

$$Q_5(z) = 2\nu\bar{\nu}zz + \alpha\nu z + \bar{\alpha}\bar{\nu}\bar{z} = 0, \quad Q_6(z) = 2\alpha\bar{\alpha}zz + \alpha\nu z + \bar{\alpha}\bar{\nu}\bar{z} = 0.$$

If z lies on either Q_5 or Q_6, then $Q_5(z)Q_6(z) = 0$. But

$$(1) \quad \frac{1}{r_s^2} - \frac{1}{r_t^2} = -(a-\bar{a})(-\alpha\nu\bar{c} + \bar{\alpha}\bar{\nu}c) - \alpha\bar{\alpha}\nu\bar{\nu}[(a-\bar{a})^2 - 2c\bar{c}] - (\alpha\nu\bar{c})^2 - (\bar{\alpha}\bar{\nu}c)^2,$$

which vanishes if and only if $r_s = r_t$. Multiplying (1) equated to zero by $-[(a-\bar{a})/(2c\bar{c})]^2$ and replacing $(a-\bar{a})/(2c)$ by m, we have $Q_5(m)Q_6(m) \equiv 0$.

THEOREM 2a. *The necessary and sufficient condition that $r_s < r_t (r_s > r_t)$ is that $z = m$ substituted in the expression for Q_5Q_6 makes that expression positive (negative).*

PROOF. $r_s \lesseqgtr r_t$ according as $1/r_s^2 - 1/r_t^2 \lesseqgtr 0$. Furthermore

$$Q_5Q_6 = -\left(\frac{1}{r_s^2} - \frac{1}{r_t^2}\right)\frac{(a-\bar{a})^2}{4c^2\bar{c}^2}.$$

* Presented at the Southwestern Section of the A.A.A.S., April, 1935.

Therefore $Q_5Q_6 \lesseqgtr 0$ according as $1/r_s^2 - 1/r_t^2 \lesseqgtr 0$, that is, according as $r_s \gtreqless r_t$.

THEOREM 2b. *The necessary and sufficient condition that $r_s < r_t$ is that m be outside both Q_5 and Q_6 or inside both; the necessary and sufficient condition that $r_s > r_t$ is that m be inside Q_5 or Q_6 and outside the other.*

PROOF. The expressions for each Q_5 and Q_6 are negative (positive) according as m is inside (outside) the circle. Theorem 2b follows from this and Theorem 2a.

REMARK 1. The diameter of Q_5 is equal to $|g| = |\bar{\alpha}/\nu|$. This can be made as large as one may wish by choosing $|\nu|$ sufficiently near zero. Furthermore, the radius of Q_6 is the reciprocal of that of Q_5. Hence, by choosing G with $|\nu|$ sufficiently near zero, one can make the region inside Q_5 or Q_6 and outside the other as nearly a half-plane as desired. Therefore, for g sufficiently large, those transformations of the group T with m in approximately one half-plane (the one g is in) have their isometric circles increased in magnitude by transforming by G whereas those with m in the other approximate half-plane have $r_s < r_t$.

Furthermore by choosing $|g|$ sufficiently near to unity one can make the region inside Q_5 or Q_6 and outside the other as small as he may wish. Thus the transformations with m in as nearly the entire plane as desired have their isometric circles decreased in magnitude by transforming by G.

THEOREM 3. *The necessary and sufficient condition that $r_s = r_t/k$, k a non-negative real number, is that m lie on the locus*

$$(2)\quad (2\alpha\bar{\alpha}\nu\bar{\nu}z\bar{z} + \alpha\nu^2\bar{\nu}z + \alpha\bar{\alpha}^2\bar{\nu}\bar{z})(2\alpha\bar{\alpha}\nu\bar{\nu}z\bar{z} + \bar{\alpha}\nu\bar{\nu}^2\bar{z} + \alpha^2\bar{\alpha}\nu z) = k^2 z\bar{z}\alpha\bar{\alpha}\nu\bar{\nu}.$$

PROOF. From the definitions of r_s and r_t and from the equation $r_s = r_t/k$, we have $(r_t/r_s)^2 = (C\overline{C})/(c\bar{c}) = k^2$. Replacing $C\overline{C}$ by its value in terms of the coefficients of T and G and then replacing $(a-\bar{a})/(2c)$ by m, we have (2), since $c/\bar{c} = -\bar{m}/m$.

REMARK 2. The number k is not determined by (2) for a real, since then $m = 0$. However, m is on both Q_5 and Q_6 for $m = 0$, and therefore, by Theorem 1, $k = 1$.

COROLLARY 1. The absolute minimum value of k is zero; this value is taken on if the midpoint of the line segments (g_i, g_i') and $(g, 1/\bar{g})$ coincide and is possible only for T an elliptic transformation.

PROOF. Substituting $m=-(\alpha\bar{\alpha}+\nu\bar{\nu})/(2\alpha\nu)$ into (2), we see that $k=0$ if $(a-\bar{a})/(2c)=-(\alpha\bar{\alpha}+\nu\bar{\nu})/(2\alpha\nu)$. Furthermore, we have $Q_0[-(\alpha\bar{\alpha}+\alpha\bar{\nu})/(2\alpha\nu)]>0$ for all G and all T of Fuchsian type, whereas $Q_0[(a-\bar{a})/(2c)]>0$ for T elliptic only.

REMARK 3. Changing (2) to trigonometric form, one finds the discriminant of the resulting quadratic in ρ to be

$$f(k) = 4(\alpha\nu e^{i\theta} + \bar{\alpha}\bar{\nu}e^{-i\theta})^2 - 16\alpha\bar{\alpha}\nu\bar{\nu}(1 - k^2).$$

This is a perfect square if and only if $k=1$ or 0; hence (2) is factorable rationally in terms of the coefficients of G in these two cases and only in them. The factors for $k=1$ are Q_5 and Q_6 of Theorem 1, and for $k=0$ they are immediate from (2).

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THE EQUATION $2^x-3^y=d$*

BY AARON HERSCHFELD

1. *Introduction.* According to Dickson's *History of the Theory of Numbers*,† Leo Hebreus, or Levi Ben Gerson (1288–1344), proved that $3^m\pm1\neq2^n$ if $m>2$, by showing that $3^m\pm1$ has an odd prime factor. The problem had been proposed to him by Philipp von Vitry in the following form: All powers of 2 and 3 differ by more than unity except the pairs 1 and 2, 2 and 3, 3 and 4, 8 and 9. In 1923 an elegant short proof by Philip Franklin appeared in the American Mathematical Monthly.‡

In 1918 G. Polya§ published a very general theorem which, as was later pointed out by S. Sivasankaranarayana Pillai,‖ proved as special cases that the equations

* Presented to the Society, October 26, 1935.

† Vol. 2, p. 731; see J. Carlebach, Dissertation, Heidelberg, 1909, pp. 62–64.

‡ Vol. 30 (1923), p. 81, problem 2927.

§ *Zur Arithmetische Untersuchung der Polynome*, Mathematische Zeitschrift, vol. 1 (1918), pp. 143–148.

‖ Journal of the Indian Mathematical Society, vol. 19 (1931), pp. 1–11.

$$a^x - b^y = d, \tag{1}$$

$$2^x - 3^y = d, \tag{2}$$

where a and b are fixed positive integers and $d \neq 0$, have at most a finite number of solutions in positive integers (x, y). Moreover, Pillai gave an asymptotic formula for the finite number of solutions of the inequality $0 < a^x - b^y \leqq n$, where $\log a/\log b$ is not rational.

In this paper we shall prove that if $|d|$ is sufficiently great, equation (2), for fixed d, can have at most one solution, while equation (1) can have no more than nine solutions.

The general existence theorems tell us in particular that the inequality $|2^x - 3^y| \leqq n$ holds for only a finite number of pairs of positive integers (x, y). But they do not tell us the precise values of such pairs (x, y) nor exactly how many exist. We shall answer these questions for $n = 10$ by solving the six equations $2^x - 3^y = d$, $d = \pm 1, \pm 5, \pm 7$, since we may obviously exclude $d = 0, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 9, \pm 10$. Similar methods apply to greater values, and we summarize some results for $n = 100$.

2. *Special Results.* If $2^x - 3^y = d$ and $x \geqq 3$, then $3^y \equiv -d \pmod 8$. But $3^n \equiv 1$ or $3 \pmod 8$. Hence $d \equiv 5$ or $d \equiv 7 \pmod 8$. Consequently for $d = 1, -5, -7$ there are no solutions of equation (1) such that $x \geqq 3$. We see therefore that these equations have only the solutions

$$2^2 - 3 = 1, \qquad 2^3 - 3^2 = -5, \qquad 2 - 3^2 = -7.$$

There remain the cases $d = -1, 5, 7$. Suppose $2^x - 3^y = -1$, so that $3^y \equiv 1 \pmod{2^x}$. But if $x > 2$, 3 belongs* to 2^{x-2} modulo 2^x, that is, 2^{x-2} is the least integer e such that $3^e \equiv 1$ modulo 2^x. Therefore $y \geqq 2^{x-2}$, if $x > 2$, and so

$$3^y - 1 \geqq 3^{2^{x-2}} - 1 \geqq 3^x - 1 > 2^x,$$

if $x \geqq 4$. Consequently $2^x - 3^y = -1$ is impossible if $x \geqq 4$. Thus the only solutions are

$$2 - 3 = -1 = 2^3 - 3^2.$$

Next consider the equation $2^x - 3^y = 5$. Obviously we have as two solutions $(x, y) = (3, 1)$ and $(5, 3)$. Suppose another solu-

* See Dickson, *Introduction to the Theory of Numbers*, pp. 16, 19.

tion (x, y) exists satisfying $2^x=3^y+5$ with $y>3$, and consequently $x>5$. Taking congruences modulo 2^4, $3^y\equiv -5 \pmod{2^5}$. But $3^3=2^5-5\equiv -5 \pmod{2^5}$ and 3 belongs to 8 modulo 2^5. Therefore $y=3+8k$, where $k>0$. Now take congruences modulo 2^6; since $x>5$, $3^y\equiv -5 \pmod{2^6}$. But $3^3\not\equiv -5 \pmod{2^6}$ and 3 belongs to 16 (mod 2^6). Thus k cannot be even and it follows that $y=11+16k'$.

Finally we take congruences modulo 17, noting that 3 belongs to 16 (mod 17), so that $2^x=3^y+5\equiv -10+5\equiv -5 \pmod{17}$. But we can easily verify that no power of 2 is congruent to -5 (mod 17). Hence we have a contradiction.

Lastly consider $2^x-3^y=7$, where $y>3$, so that $x>2$. Taking congruences modulo 3 and then modulo 4, we see that both x and y are even. Therefore

$$1 \leqq 2^{x/2} - 3^{y/2} = 7/(2^{x/2} + 3^{y/2}) < 1,$$

which is a contradiction. Hence the only solution is $2^4-3^2=7$.

We may summarize by tabulating our results:

d	(x, y)	d	(x, y)
-1	$(1, 1)$; $(3, 2)$	1	$(2, 1)$
-5	$(2, 2)$	5	$(3, 1)$; $(5, 3)$
-7	$(1, 2)$	7	$(4, 2)$

3. *General Theorems.* We have now shown that when $x>5$, or $y>3$, then $|2^x-3^y|>10$. The author has verified in an unpublished paper that similar methods may be used to prove that if x and y are positive integers such that $x>8$ or $y>5$, then $|2^x-3^y|>100$. In no case were there found more than two solutions of the equation (2) for any fixed d with $|d|\leqq 100$. We conclude by proving that *if $|d|$ is sufficiently great, the equation $2^x-3^y=d$ cannot have more than one solution.*

We shall use the property proved by S. S. Pillai in his 1931 paper, previously mentioned, that given any $\delta>0$ and integers a and b such that $\log a/\log b$ is not rational, there exists an integer $x_1=x_1(\delta)$ such that for all $x>x_1$ and all positive (a^x-b^y),

$$0<a^x-b^y>a^{x(1-\delta)}.$$

Suppose that $2^x-3^y=2^X-3^Y=d$, $X>x$, $Y>y$. Then

$$2^X - 2^x = 3^Y - 3^y, \qquad 2^x(2^{X-x} - 1) = 3^y(3^{Y-y} - 1),$$

$$2^{X-x} \equiv 1 \pmod{3^y}, \qquad 3^{Y-y} \equiv 1 \pmod{2^x},$$

$$X - x \geqq 2 \cdot 3^{y-1}, \quad X \geqq 2 \cdot 3^{y-1}, \quad Y - y \geqq 2^{x-2}, \quad Y \geqq 2^{x-2},$$

if $x > 2$.

Consider first $d > 0$. Choose any positive $\delta < 1/2$ and let $x_1 = x_1(\delta)$, where for all $x > x_1$ and $0 < 2^x - 3^y$, we have $0 < 2^x - 3^y > 2^{x(1-\delta)} > 2^{x/2}$. Let us consider only the positive values of d greater than 2^{x_1+5}. Then

$$2^x > d > 2^{x_1+5}, \qquad x > x_1 + 5, \qquad X > x > x_1,$$

$$2^X = 3^Y + d > 3^Y \geqq 3^{2^{x-2}} > 2^{2^{x-2}}, \qquad X > 2^{x-2} > 2x,$$

since $x > 5$. Therefore $d = 2^X - 3^Y > 2^{X/2} > 2^x > d$, a contradiction.

Next, for $d < 0$, consider only values of d such that $|d| > 3^{y_1+2}$, where $y_1 = y_1(\delta)$, so that for all $y > y_1$ and $0 < 3^y - 2^x$, we have $0 < 3^y - 2^x > 3^{y(1-\delta)} > 3^{y/2}$. Hence, if (2) has two solutions,

$$3^y > |d| > 3^{y_1+2}, \qquad y > y_1 + 2, \qquad Y > y > y_1,$$

$$3^Y = 2^X - d > 2^X \geqq 2^{2 \cdot 3^{y-1}} > 3^{3^{y-1}}, \qquad Y > 3^{y-1} > 2y,$$

since $y > 2$. Thus $-d = 3^Y - 2^X > 3^{Y/2} > 3^y > -d$, a contradiction.

In conclusion we can say of the general equation $a^x - b^y = d$, that if $|d|$ is sufficiently great, this equation can have at most nine solutions. This is a simple consequence of the theorem due to C. L. Siegel,* that $ax^n - by^n = k$ (fixed $n \geqq 3$) has at most one solution if $|ab|$ is sufficiently great. For if $|d|$ is large enough we can write equation (1) in the form of at least one of nine equations

$$a^i A u^2 - b^j B v^2 = d, \qquad (i, j = 0, 1, 2),$$

where $|AB|$ is so great that each of the nine equations must, by Siegel's theorem, have at most one solution.

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* Abhandlungen Akademie Berlin, 1929, Nr. 1, 70 pp.

SOME THEOREMS ON ABSTRACT SPACES

BY R. E. A. C. PALEY*

1. *Introduction.* We shall solve some of the questions which are left unanswered by Banach in his treatise *Théorie des Opérations Linéaires* (Warszawa, 1932). They are taken from Chapter 12, pp. 193–207, and we shall assume throughout this paper that the reader is familiar with the ideas and results of that section.

We denote by $l^{(k)}$ the set of sequences $\{a_1, a_2, \cdots\}$ such that $\sum |a_n|^k < \infty$, and we denote by $L^{(k)}$ the set of measurable functions $f(x)$ defined on the interval $(0, 1)$ for which

$$\int_0^1 |f(x)|^k dx < \infty .$$

Suppose that A and B are two vectorial spaces. We say that $\dim_l(A) \geqq \dim_l(B)$, if B is isomorphic with a vectorial subspace of A, and that $\dim_l(A) = \dim_l(B)$, if $\dim_l(A) \geqq \dim_l(B)$ and $\dim_l(B) \geqq \dim_l(A)$. Thirdly, we say that $\dim_l(A) > \dim_l(B)$, if $\dim_l(A) \geqq \dim_l(B)$ and if it is not true that $\dim_l(A) = \dim_l(B)$.

Banach has shown (loc. cit.) that if $\dim_l(l^{(q)}) \geqq \dim_l(L^{(p)})$ for $q > 1 < p$, then $q = p = 2$, and that $\dim_l(L^{(p)}) \geqq \dim_l(l^{(p)})$, $p > 1$, where the equality sign occurs only when $p = 2$. He has also shown that the relation $\dim_l(l^{(q)}) \leqq \dim_l(L^{(p)})$ is impossible if $q > p > 2$ and $q < p < 2$. We shall show that this relation is also impossible in the cases† (i) $p > 2 > q$, (ii) $q > 2 > p$, (iii) $2 < q < p$.

2. *Certain Inequalities.* Let us suppose that $\dim_l(l^{(q)}) \leqq \dim_l(L^{(p)})$, $(q > 1 < p)$. Then $l^{(q)}$ is isomorphic with a vectorial closed subspace of $L^{(p)}$. Suppose that to the sequences, $(1, 0, 0, \cdots)$, $(0, 1, 0, \cdots)$, $(0, 0, 1, 0, \cdots)$, $\cdots$ of $l^{(q)}$ correspond functions, $f_1(x)$, $f_2(x)$, $f_3(x)$, $\cdots$ of $L^{(p)}$. Then to the sequence $(a_1, a_2, \cdots, a_r, 0, 0, \cdots)$ corresponds

* This note was found, in rather incomplete form, among the papers of the late R. E. A. C. Paley. Various modifications and corrections were needed in order to bring it to its present form, but an effort was made to keep as close to the original as possible. F. J. MURRAY.

† We were not able to reconstruct the proof in the remaining case $p < q < 2$, from the fragments contained in the manuscript. F.J.M.

the function $a_1f_1(x)+a_2f_2(x)+\cdots+a_rf_r(x)$. Since the two spaces are isomorphic, it follows that the operation and its inverse are both limited and thus (see Banach, loc. cit., Theorem I, p. 54) the ratio

$$\left\{\int_0^1\left|\sum_{m=1}^{n}a_mf_m(x)\right|^p dx\right\}^{1/p}\Big/\left\{\sum_{m=1}^{n}|a_m|^q\right\}^{1/q}$$

must lie between two positive constants.

Let $\phi_1(t), \phi_2(t), \cdots$ denote Rademacher functions.* We use the following inequality due to Khintchine.† We have, for $k>0$,

$$(1)\quad B_k \leqq \left(\int_0^1\left|\sum a_n\phi_n(t)\right|^k dt\right)\Big/\left(\sum|a_n|^2\right)^{k/2} \leqq B_k;$$

here and in the sequel B denotes a positive absolute constant, B_k denotes one which depends only on k, and so on ($B, B_k, \cdots$ may denote different constants in different contexts).

Now suppose that

$$B_{p,q} \leqq \left\{\int_0^1\left|\sum a_nf_n(x)\right|^p dx\right\}^{1/p}\Big/\left\{\sum|a_n|^q\right\}^{1/q} \leqq B_{p,q}.$$

Then we have, for every t,

$$B_{p,q} \leqq \left\{\int_0^1\left|\sum a_n\phi_n(t)f_n(x)\right|^p dx\right\}^{1/p}\Big/\left\{\sum|a_n|^q\right\}^{1/q} \leqq B_{p,q},$$

and thus raising to the pth power, integrating with respect to t, using (1), and then extracting the pth root, we find

$$(2)\quad B_{p,q} \leqq \frac{\left[\int_0^1\left\{\sum|a_n|^2|f_n(x)|^2\right\}^{p/2}dx\right]^{1/p}}{\left\{\sum|a_n|^q\right\}^{1/q}} \leqq B_{p,q}.$$

We observe that if we take $a_m=1$, $a_n=0$, $n\neq m$, then (2) shows that

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* See Rademacher 4, Khintchine 2, Paley and Zygmund 3, Kaczmarz and Steinhaus 1.

† Khintchine 2, Paley and Zygmund 3.

$$B_{p,q} \leqq \left[\int_0^1 |f_m(x)|^p dx\right]^{1/p} \leqq B_{p,q}.$$

3. *Proof for Cases* (i) *and* (ii). First suppose that $p>2>q$. Then

$$\begin{aligned}
\{\sum |a_n|^q\}^{1/q} &\leqq B_{p,q}\left[\int_0^1 \{\sum |a_n|^2 |f_n(x)|^2\}^{p/2} dx\right]^{1/p} \\
&= B_{p,q}\left(\left[\int_0^1 \{\sum |a_n|^2 |f_n(x)|^2\}^{p/2} dx\right]^{2/p}\right)^{1/2} \\
&\leqq B_{p,q}\left(\sum\left[\int_0^1 \{|a_n|^2 |f_n(x)|^2\}^{p/2} dx\right]^{2/p}\right)^{1/2} \\
&= B_{p,q}\left(\sum |a_n|^2 \left[\int_0^1 |f_n(x)|^p dx\right]^{2/p}\right)^{1/2} \\
&\leqq B_{p,q}[\sum |a_n|^2]^{1/2},
\end{aligned}$$

which is of course impossible.

Suppose that $q>2>p$. Then

$$\begin{aligned}
\{\sum |a_n|^q\}^{1/q} &\geqq B_{p,q}\left[\int_0^1 \{\sum |a_n|^2 |f_n(x)|^2\}^{p/2} dx\right]^{1/p} \\
&= B_{p,q}\left[(\sum_m |a_m|^2)^{p/2}\right. \\
&\quad \left.\cdot \int_0^1 \{\textstyle\sum_n (|a_n|^2 / (\sum_m |a_m|^2)) |f_n(x)|^2\}^{p/2} dx\right]^{1/p} \\
&\geqq B_{p,q}\left[(\textstyle\sum_m |a_m|^2)^{p/2} \sum_n (|a_n|^2 / (\sum_m |a_m|^2)) \int_0^1 |f_n(x)|^p dx\right]^{1/p} \\
&\geqq B_{p,q}(\textstyle\sum |a_n|^2)^{1/2}.
\end{aligned}$$

This too is impossible.

4. *Proof for Case* (iii). Suppose that $2<q<p$. Let

$$\sum_{n=2}^{\infty} |a_n|^q = A^q.$$

Then

$$\int_0^1 |f_1(x)|^{p-2}\Big(\sum_2^\infty |a_n|^2 |f_n(x)|^2\Big)dx$$

$$= A^{2-p}\int_0^1 (A^2|f_1(x)|^2)^{p/2-1}\Big(\sum_2^\infty |a_n|^2 |f_n(x)|^2\Big)dx$$

$$\leqq A^{2-p}\int_0^1 \Big\{A^2|f_1(x)|^2 + \sum_2^\infty |a_n|^2 |f_n(x)|^2\Big\}^{p/2} dx \leqq B_{p,q}A^2.$$

Thus we have

$$\sum_2^\infty |a_n|^2 \int_0^1 |f_1(x)|^{p-2}|f_n(x)|^2 dx \leqq B_{p,q}\Big(\sum_2^\infty |a_n|^q\Big)^{2/q},$$

and so it follows that

$$\int_0^1 |f_1(x)^{p-2}|f_n(x)|^2 dx \to 0 \text{ as } n \to \infty.$$

Let N be a large finite number. Choose $n_1 = 1, n_2, n_3, \cdots, n_N$, so that

$$\int_0^1 |f_1(x)|^{p-2}|f_{n_k}(x)|^2 dx \leqq \epsilon^{-N}, \qquad (k = 2, 3, \cdots, N),$$

$$\int_0^1 |f_{n_2}(x)|^{p-2}|f_{n_k}(x)|^2 dx \leqq \epsilon^{-N}, \qquad (k = 3, 4, \cdots, N),$$

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and so on. We set $f_{n_k}(x) = g_k(x)$ and consider the integral

$$(3) \qquad \int_0^1 \Big(\sum_{k=1}^N |f_{n_k}(x)|^2\Big)^{p/2} dx = \int_0^1 \Big(\sum_{k=1}^N |g_k(x)|^2\Big)^{p/2} dx.$$

Let μ be the integer such that $\mu \leqq p/2 < \mu+1$. Then the last integral does not exceed

$$\sum_{k_1=1}^N \sum_{k_2=1}^N \cdots \sum_{k_{\mu+1}=1}^N \int_0^1 |g_{k_1}(x)|^2 |g_{k_2}(x)|^2 \cdots |g_{k_{\mu+1}}(x)|^{p-2\mu} dx$$

$$\leqq \sum_{k=1}^N \int_0^1 |g_k(x)|^p dx$$

$$+ N^{\mu+1} \max_{k_1,\cdots,k_{\mu+1} \text{ not all equal}} \int_0^1 |g_{k_1}(x)|^2 \cdots |g_{k_{\mu+1}}(x)|^{p-2\mu} dx.$$

Now

$$\int_0^1 |g_{k_1}(x)|^2 \cdots |g_{k_{\mu+1}}(x)|^{p-2\mu} dx$$

$$\leqq \left\{\int_0^1 |g_{k_1}(x)|^2 |g_{k_2}(x)|^{p-2} dx\right\}^{2/(p-2)}$$

$$\cdot \left\{\int_0^1 |g_{k_1}(x)|^2 |g_{k_3}(x)|^{p-2} dx\right\}^{2/(p-2)} \cdots$$

$$\cdot \left\{\int_0^1 |g_{k_1}(x)|^2 |g_{k_{\mu+1}}(x)|^{p-2} dx\right\}^{(p-2\mu)/(p-2)},$$

and if $k_1, k_2, \cdots, k_{\mu+1}$ are not all equal, k_1 must be different from one of the $k_2, k_3, \cdots, k_{\mu+1}$, and since each of the integrals in the above product is certainly bounded by a constant of the form $B_{p,q}$, it follows that the last expression does not exceed an expression of the form

$$B_{p,q}\left\{\int_0^1 |g_k(x)|^2 |g_{k'}(x)|^{p-2} dx\right\}^{(p-2\mu)/(p-2)},$$

where $k \neq k'$. If now $k < k'$, then

$$\int_0^1 |g_k(x)|^2 |g_{k'}(x)|^{p-2} dx \leqq e^{-N},$$

while if $k > k'$, $p \geqq 4$,

$$\int_0^1 |g_k(x)|^2 |g_{k'}(x)|^{p-2} dx$$

$$\leqq \left\{\int_0^1 |g_k(x)|^{p-2} |g_{k'}(x)|^2 dx\right\}^{2/(p-2)} \left\{\int_0^1 |g_{k'}(x)|^p dx\right\}^{(p-4)/(p-2)}$$

$$\leqq B_{p,q} e^{-2N/(p-2)},$$

and if $p \leqq 4$,

$$\int_0^1 |g_k(x)|^2 |g_{k'}(x)|^{p-2} dx$$

$$\leqq \left\{\int_0^1 |g_k(x)|^{p-2} |g_{k'}(x)|^2 dx\right\}^{(p-2)/2} \left\{\int_0^1 |g_k(x)|^p dx\right\}^{(4-p)/2}$$

$$\leqq B_{p,q} e^{-(p-2)N/2}.$$

Hence the expression (3) is less than an expression of the form

$$B_{p,q}\{N + N^{\mu+1}(e^{-N} + e^{-2N/(p-2)} + e^{-(p-2)N/2)(p-2\mu)/(p-2)}\} \leqq B_{p,q}N,$$

if N is chosen sufficiently large. This contradicts the hypothesis

$$\int_0^1 \left\{ \sum_{k=1}^{N} |f_{n_k}(x)|^2 \right\}^{p/2} dx \geqq B_{p,q}N^{p/q},$$

and we have shown that (iii) is impossible.

5. *Conclusion.* In conclusion it might be pointed out that if the relation $\dim_l(l^{(q)}) \leqq \dim_l(L^{(p)})$ is impossible, then also the relation $\dim_l(L^{(q)}) \leqq \dim_l(L^{(p)})$ is impossible. For if $\dim_l(L^{(q)}) \leqq \dim_l(L^{(p)})$, then since $\dim_l(l^{(q)}) < \dim_l(L^{(q)})$, it follows that $\dim_l(l^{(q)}) \leqq \dim_l(L^{(p)})$, a contradiction. Thus in each of the Cases (i), (ii), and (iii), we also have the statement of the impossibility of the relationship $\dim_l(L^{(q)}) \leqq \dim_l(L^{(p)})$. For Cases (i) and (ii), these last results are given by Banach (loc. cit.); however Case (iii) is new.

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3. R. E. A. C. Paley and A. Zygmund, *On some series of functions*, Proceedings of the Cambridge Philosophical Society, vol. 26 (1930), pp. 337–357.

4. H. Rademacher, *Einige Sätze über Reihen von allgemeine Orthogonalfunktionen*, Mathematische Annalen, vol. 87 (1922), pp. 112–138.

ON THE UNIVALENCY OF CESÀRO SUMS OF UNIVALENT FUNCTIONS*

BY M. S. ROBERTSON†

Introduction. Let

$$f(z) = z + \sum_{2}^{\infty} a_n z^n \tag{1}$$

be analytic and univalent for $|z| < 1$, that is, if z_1 and z_2 $(z_1 \neq z_2)$ are any two points inside the unit circle, then

$$f(z_1) - f(z_2) \neq 0.$$

The partial sums $S_n(z)$ of $f(z)$,

$$S_n(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n, \tag{2}$$

in general are not univalent in the unit circle, though they are univalent for $|z| < 1/4$ as G. Szegö has shown.‡ In some cases, however, one can say that the partial sums are univalent in the whole unit circle. J. W. Alexander§ has shown, for example, that if the coefficients of (1) are real and positive and such that the numbers na_n form a decreasing sequence, then not only $f(z)$ but all its partial sums are univalent in the unit circle.

To the best of the writer's knowledge the only results obtained to date regarding the univalency of the Cesàro sums of univalent functions are those of L. Fejér‖ who showed that if $f(z)$ is real on the real axis, and convex in the direction of the imaginary axis for $|z| < 1$, then all the Cesàro sums of the third order are univalent for $|z| < 1$.

In this paper we show that if the ordinary partial sums of

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† National Research Fellow.

‡ See G. Szegö, *Zur Theorie der schlichten Abbildungen*, Mathematische Annalen, vol. 100 (1928), pp. 188–211.

§ See J. W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Annals of Mathematics, (2), vol. 17 (1915–16), pp. 12–22.

‖ See L. Fejér, *Neue Eigenschaften der Mittelwerte bei den Fourierreihen*, Journal of the London Mathematical Society, vol. 8 (1933), pp. 53–62.

$f(x)$ are univalent for $|z|<1$ when $f(z)$ has this property, then also all the Cesàro sums of the first order are univalent in the unit circle.

For the proof of our theorem we make use of the following theorem due to N. Obrechkoff* which we submit as the lemma below.

LEMMA. *Let $P(z)$ be a polynomial of degree n whose zeros are all outside $|z|=1$, and let $Q(z)$ be a polynomial of degree m whose zeros are inside $|z|\leqq r<1$. If λ is an arbitrary positive number, then the polynomial*

$$2z[P'(z)\cdot Q(z)-\lambda P(z)\cdot Q'(z)]+(m\lambda-n)P(z)\cdot Q(z)$$

has no zeros in the annulus $r<|z|<1$.

In particular, if we take $Q(z)\equiv z$, then the polynomial $2z[zP'(z)-\lambda P(z)]+(\lambda-n)z\cdot P(z)$ has no zeros in $0<|z|<1$. In other words, $[P(z)-(2/(\lambda+n))zP'(z)]$ has no zeros in $|z|<1$.

Let

$$S_n(z)=z+a_2z^2+a_3z^3+\cdots+a_nz^n \tag{3}$$

be univalent in the unit circle. The necessary and sufficient condition for $S_n(z)$ to be univalent for $|z|<1$ is, as J. Dieudonné has shown,† that the zeros of the polynomial

$$P(z,\theta)\equiv 1+\sum_{k=2}^{n}a_k\frac{\sin k\theta}{\sin\theta}z^{k-1} \tag{4}$$

lie outside of the unit circle for all values of θ. If $S_n(z)$ is univalent for $|z|<1$, we know by our lemma that $[P(z,\theta)-(2/(\lambda+n-1))zP'(z,\theta)]$ has no zeros in $|z|<1$ or that

$$F(z,\theta)\equiv 1+\sum_{k=2}^{n}\left\{1-\frac{2(k-1)}{\lambda+n-1}\right\}a_k\frac{\sin k\theta}{\sin\theta}z^{k-1} \tag{5}$$

has no zeros inside $|z|<1$ for all θ. It follows by the theorem of Dieudonné, then, that the polynomial

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* See N. Obrechkoff, *Sur les racines des équations algébriques*, Tôhoku Mathematical Journal, vol. 38 (1933), p. 100.

† See J. Dieudonné, Annales de l'Ecole Normale, vol. 48 (1931), p. 310.

$$(6) \qquad f_n(z, \lambda) \equiv z + \sum_{k=2}^{n} \left\{1 - \frac{2(k-1)}{\lambda + n - 1}\right\} a_k z^k$$

is univalent in the unit circle for all $\lambda > 0$.

We denote the nth Cesàro sum of the first order of (3) by

$$(7) \qquad S_n^{(1)}(z) \equiv \frac{S_1(z) + S_2(z) + \cdots + S_n(z)}{n},$$

$$(8) \qquad \begin{aligned} S_n^{(1)}(z) &= z + \left(1 - \frac{1}{n}\right) a_2 z^2 + \cdots \\ &\quad + \left(1 - \frac{k-1}{n}\right) a_k z^k + \cdots + \frac{a_n}{n} z^n. \end{aligned}$$

If in (6) we give λ successively the values $n-1$, $n+1$, $n+3$, $\cdots$, we then see that the Cesàro sums of $S_n(z)$

$$S_{n-1}^{(1)}(z),\ S_n^{(1)}(z),\ S_{n+1}^{(1)}(z), \cdots$$

are univalent for $|z| < 1$ whenever $S_n(z)$ has this property. Thus we have the following theorem.

THEOREM. *If the polynomial*

$$S_n(z) = z + a_2 z^2 + \cdots + a_n z^n$$

is univalent in the unit circle, then the $(n+k)$th Cesàro partial sum of the first order of $S_n(z)$ is also univalent in the unit circle for $k \geqq -1$.

COROLLARY. *If all the partial sums $S_n(z)$ of $f(z)$ are univalent in the unit circle, then so also are all the Cesàro sums of the first order of $f(z)$.*

YALE UNIVERSITY

A RECURSION FORMULA FOR THE POLYNOMIAL SOLUTIONS OF A PARTIAL DIFFERENTIAL EQUATION*

BY C. C. MacDUFFEE

Let us denote by O the differential operator

$$\frac{\partial^n}{\partial x^n} - a_1\frac{\partial^n}{\partial x^{n-1}\partial y} - a_2\frac{\partial^n}{\partial x^{n-2}\partial y^2} - \cdots - a_n\frac{\partial^n}{\partial y^n},$$

where the a's are constants, real or complex. Define

$$\phi(\lambda) = \lambda^n - a_1\lambda^{n-1} - a_2\lambda^{n-2} - \cdots - a_n.$$

Let A be any nth order matrix having $\phi(\lambda) = 0$ as its minimum equation, and define $Z = Ax + Iy$. For any function $F(Z)$,

$$\frac{\partial^n F}{\partial x^{n-t}\partial y^t} = \frac{d^n F}{dZ^n}A^{n-t}.$$

Then

$$OF = \frac{d^n F}{dZ^n}\phi(A),$$

and since $\phi(A) = 0$, $F(Z)$ is a solution of $OF = 0$. If we can write

$$F(Z) = f_1(x, y)I + f_2(x, y)A + \cdots + f_n(x, y)A^{n-1},$$

we have

$$OF(Z) = \sum_{i=1}^{n} Of_i(x, y)A^{i-1} = 0.$$

Since A satisfies no equation of degree $n-1$,

$$Of_i(x, y) = 0 \qquad (i = 1, 2, \cdots, n).$$

These results are due to Spampinato.†

By a proper choice of the matrix A, this theorem of Spampinato leads to an interesting recursion formula for the poly-

* Presented to the Society, November 30, 1935.

† *Atti Accademia Lincei, Rendiconti*, (6), vol. 21 (1935), pp. 73–76.

nomial solutions of $Of(x, y)=0$ by means of which all such solutions can be computed with rapidity.

Let

$$A = \begin{Vmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{Vmatrix}.$$

Then $\phi(\lambda)=0$ is the minimum equation of A. By means of the relation $\phi(A)=0$, $(Ax+Iy)^k$ can be written as a linear combination of $I, A, A^2, \cdots, A^{n-1}$. The first rows of these matrices are, respectively,

$$(1, 0, 0, \cdots, 0), (0, 1, 0, \cdots, 0), (0, 0, 1, \cdots, 0), \cdots, (0, 0, 0, \cdots, 1).$$

Hence if we denote the first row of $(Ax+Iy)^k$ by $p_1^{(k)}, p_2^{(k)}, \cdots, p_n^{(k)}$, we have

$$(Ax + Iy)^k = p_1^{(k)} I + p_2^{(k)} A + \cdots + p_n^{(k)} A^{n-1}.$$

If we define the vector

$$v_k = (p_1^{(k)}, p_2^{(k)}, \cdots, p_n^{(k)}),$$

it follows from the relation

$$(Ax + Iy)^{k+1} = (Ax + Iy)^k (Ax + Iy),$$

that

$$v_{k+1} = v_k (Ax + Iy),$$

or in full,

$$\begin{aligned}
p_1^{(k+1)} &= yp_1^{(k)} && && + xa_n\, p_n^{(k)}, \\
p_2^{(k+1)} &= xp_1^{(k)} + yp_2^{(k)} && && + xa_{n-1} p_n^{(k)}, \\
p_3^{(k+1)} &= && xp_2^{(k)} + yp_3^{(k)} && + xa_{n-2} p_n^{(k)}, \\
&\cdots\cdots\cdots

It is evident that $p_i^{(k)}$ is a homogeneous polynomial of degree k, and by Spampinato's theorem, it is a solution of $Of=0$.

For $k \geqq n$ there are just n linearly independent homogeneous polynomials of degree k which satisfy $Of=0$. For if we let

$$P_k = A_0x^k + A_1x^{k-1}y + \cdots + A_ky^k$$

where the A's are to be determined, then $\partial^n P_k/\partial x^n$ will have the form

$$h_1A_0x^{k-n} + h_2A_1x^{k-n-1}y + \cdots + h_kA_{k-n}y^{k-n},$$

where each h is a positive integer. By forming OP_k, we obtain a homogeneous polynomial of degree $k-n$ which must vanish identically. This condition leads to $k-n+1$ equations in the $k+1$ unknown A's. Since the matrix of the coefficients

$$\begin{Vmatrix} h_1 & * & * & \cdots \\ 0 & h_2 & * & \cdots \\ 0 & 0 & h_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{Vmatrix}$$

is obviously of rank $k-n+1$, there are exactly n linearly independent solutions.

For $k<n$, the components of v_k are, in order, the terms in the binomial expansion of $(y+x)^k$ followed by zeros. Hence, for $k<n$, every homogeneous polynomial of degree k is a linear combination of $p_1^{(k)}, p_2^{(k)}, \cdots, p_n^{(k)}$.

We shall prove by induction that for $k \geqq n-1$,

$$(2) \qquad p_i^{(k)} = c_{ik}y^{k-i+1}x^{i-1} + \cdots, \qquad (i = 1, 2, \cdots, n),$$

where the c_{ik} are positive integers and the terms omitted are of degree less than $k-i+1$ in y. For $k=n-1$, (2) follows from (1). If we assume (2) to hold for some $k \geqq n-1$, we have from (1)

$$\begin{aligned} p_i^{(k+1)} &= xp_{i-1}^{(k)} + yp_i^{(k)} + xa_{n-i+1}p_n^{(k)} \qquad (p_0^{(k)} = 0) \\ &= xc_{i-1,k}y^{k-i+2}x^{i-2} + yc_{ik}y^{k-i+1}x^{i-1} \\ &\quad + xa_{n-i+1}c_{nk}y^{k-n+1}x^{n-1} + \cdots \\ &= (c_{i-1,k} + c_{ik})y^{k-i+2}x^{i-1} + \cdots, \end{aligned}$$

where the terms represented by $\cdots$ are of degree less than $k-i+2$ in y. Thus (2) is established for $k+1$ and hence by induction for $k \geqq n-1$. Hence $p_1^{(k)}, p_2^{(k)}, \cdots, p_n^{(k)}$ are linearly independent for $k \geqq n-1$.

We have now proved that, for every k, every linear homogeneous polynomial of degree k which is a solution of $Of=0$ has the form

$$c_1 p_1^{(k)} + c_2 p_2^{(k)} + \cdots + c_n p_n^{(k)},$$

where the c's are arbitrary constants.

THE UNIVERSITY OF WISCONSIN

A CHARACTERIZATION OF NULL SYSTEMS IN PROJECTIVE SPACE

BY RICHARD BRAUER

1. *Introduction.* We consider the n-dimensional projective space S_n defined analytically by means of any abstract field F. The points P of S_n are given by a set of $n+1$ elements x_i of F, $P=(x_0, x_1, \cdots, x_n)$, (not all $x_i=0$), with the convention that proportional sets define the same point. The points P whose coordinates satisfy a linear homogeneous equation $u^{(0)}x_0+u^{(1)}x_1+\cdots+u^{(n)}x_n=0$, (not all $u^{(i)}=0$), form a hyperplane $\epsilon=(u^{(0)}, u^{(1)}, \cdots, u^{(n)})$. There is no difficulty in defining such notions as those of straight lines, projections, and cross ratios, and discussing the elementary properties.

Let M be a non-singular skew-symmetric bilinear form with coefficients a_{ik} in F,

$$M = \sum_{i,k=0}^{n} a_{ik} y_i x_k, \qquad a_{ik} = -a_{ki}, \qquad \det(a_{ik}) \neq 0.$$

For every point $P=(x_0, x_1, \cdots, x_n)$ the equation $M=0$ is the equation of a hyperplane ϵ in the coordinates $(y_0, y_1, \cdots, y_n)$ of a variable point of ϵ. We obtain in this manner a one-to-one correspondence between the points $P=(x_0, x_1, \cdots, x_n)$ and hyperplanes $\epsilon=(u^{(0)}, u^{(1)}, \cdots, u^{(n)})$ of S_n which is called a *null system*. The relation between corresponding values of the $u^{(i)}$ and x_i is given by

$$u^{(i)} = \sum_{k=0}^{n} a_{ik}x_k.$$

The assumption that $\det(a_{ik})$ is different from zero implies that n is odd.

As is easily seen a null system has the following properties:

(1) Every point P lies in the associated hyperplane ϵ.

(2) If P corresponds to ϵ, then the points Q in ϵ are associated with hyperplanes through P.

Conversely, *a one-to-one correspondence between points and hyperplanes of S_n with these two properties is a null system.* A geometric proof of this fact for the case $n=3$ is contained in Chapter 11, Volume 1, of the book by O. Veblen and J. W. Young on projective geometry. In this note I will give an analytic proof for any odd number of dimensions. The proof by Veblen and Young and the following one are independent of any properties of continuity. They hold for every field F. It may be remarked that the property (1) alone is not sufficient to characterize null systems.

2. *Definition of Collineation.* We assume that a one-to-one correspondence between the elements of S_n is given by

$$(1a) \qquad P = (x_0, x_1, \cdots, x_n) \to P' = (x_0', x_1', \cdots, x_n'),$$

and, furthermore, a one-to-one correspondence between the hyperplanes by

$$(1b) \qquad \begin{aligned} \epsilon &= (u^{(0)}, u^{(1)}, \cdots, u^{(n)}) \\ &\to \epsilon' = ((u')^{(0)}, (u')^{(1)}, \cdots, (u')^{(n)}). \end{aligned}$$

We call the correspondence a *collineation*, if the point P' lies in the hyperplane ϵ' when and only when P lies in ϵ.

If the mapping $\zeta \to \phi(\zeta)$ is an automorphism† of the field F, if

† An automorphism of a field F is a one-to-one mapping of the elements of a field onto themselves, $\zeta \to \zeta^* = \phi(\zeta)$, such that

$$\phi(\zeta_1 + \zeta_2) = \phi(\zeta_1) + \phi(\zeta_2), \qquad \phi(\zeta_1\zeta_2) = \phi(\zeta_1)\phi(\zeta_2)$$

for any two elements ζ_1 and ζ_2 of F. If a symbol ∞ is added to the elements of F, we postulate $\phi(\infty) = \infty$.

The zero element 0 and the unit element 1 correspond to themselves under an automorphism of any field F.

Let, for instance, F be the field of all real and complex numbers. One obtains an automorphism by associating with any ζ the conjugate complex number $\bar{\zeta}$. There are no automorphisms of the field of all real numbers except the identical automorphism $\zeta \to \zeta$.

$A=(a_{ik})$ is a non-singular matrix of degree $n+1$, $(i, k=0, 1, 2, \cdots, n)$, and if $A'^{-1}=(\alpha_{ik})$ is the contragredient matrix, the transformation

$$(2) \qquad x_i' = \sum_{k=0}^{n} a_{ik}\phi(x_k), \qquad u'^{(i)} = \sum_{k=0}^{n} \alpha_{ik}\phi(u^{(k)}),$$

defines a collineation.

3. *Equations of the General Collineation.* It is a well known theorem that *the most general collineation for $n>1$ is given by* (2). I will begin with a sketch of a proof of this theorem which seems to be more direct than the proofs previously given.

We consider a fixed collineation. It follows from the definition that points on a straight line are always transformed into points on a straight line. Let P_1, P_2, P_3, P_4 and Q_1, Q_2, Q_3, Q_4 be two sets of four points, each set lying on a straight line such that both sets have the same cross ratio λ. Then, it is possible to carry over the first set into the second one by effecting a succession of projections. We denote by R the figure consisting of all points and lines which are necessary for this construction.

We apply the given collineation to the whole figure R. If P_r is transformed into P_r' and Q_r into Q_r', the image R' of R is a figure consisting of points and lines which shows immediately that it is possible to carry P_1', P_2', P_3', P_4' over into Q_1', Q_2', Q_3', Q_4' by successive projections. Hence these sets have the same cross ratio λ'. Consequently, λ' is a function $\phi(\lambda)$ of λ alone and independent of the special choice of P_1, P_2, P_3, P_4. Hence four points on a straight line with the cross ratio λ are always transformed by the given collineation into points with the cross ratio $\lambda'=\phi(\lambda)$.

If P_1 and P_2 coincide, so do P_1' and P_2'. Hence we have $\lambda'=1$, if $\lambda=1$, or $\phi(1)=1$. Similarly, we find $\phi(0)=0, \phi(\infty)=\infty$. The points P_1, P_3, P_2, P_4 in this order correspond to P_1', P_3', P_2', P_4'. The cross ratios being $1-\lambda$ and $1-\phi(\lambda)$, we derive

$$(3) \qquad \phi(1-\lambda) = 1-\phi(\lambda).$$

Let P_5 be a point on the straight line through P_1, P_2, P_3, P_4 such that the cross ratio $P_1P_2P_4P_5$ has a given value μ. According to a simple formula in analytic geometry, the cross ratio $P_1P_2P_3P_5$ has the value $\lambda\mu$. If by the collineation P_5 is transformed into P_5', the sets P_1', P_2', P_3', P_4'; P_1', P_2', P_4', P_5';

P_1', P_2', P_3'; P_4' have the cross ratios $\phi(\lambda)$, $\phi(\mu)$, $\phi(\lambda\mu)$, respectively. But the last one is the product of the preceding ones, that is,

$$\phi(\lambda\mu) = \phi(\lambda)\phi(\mu). \tag{4}$$

Equations (3) and (4) show that the mapping $\lambda \to \phi(\lambda)$ establishes an automorphism of the field F. For, if $\lambda \neq 0$, we have

$$\begin{aligned} \phi(\lambda - \mu) &= \phi(\lambda)\phi\left(1 - \frac{\mu}{\lambda}\right) = \phi(\lambda)\left(1 - \phi\left(\frac{\mu}{\lambda}\right)\right) \\ &= \phi(\lambda) - \phi(\mu). \end{aligned} \tag{5}$$

From (4), for $\lambda = \mu = -1$, we derive $\phi(-1) = \pm 1$. If $\phi(-1) = +1$ we obtain $\phi(2) = 0$ from (3) for $\lambda = -1$ and hence $\phi(2\lambda) = 0$ from (4) for any λ. If the characteristic of F is not 2, we find a contradiction by putting $\lambda = -1/2$. In the case of characteristic 2, we have $-1 = 1$. Therefore, $\phi(-1) = -1$ and then $\phi(-\mu) = -\mu$, according to (4) for $\lambda = -1$. Hence (5) holds in the case $\lambda = 0$. By replacing μ by $-\mu$ in (5), we obtain $\phi(\lambda+\mu) = \phi(\lambda) + \phi(\mu)$. Therefore, $\lambda \to \phi(\lambda)$ is really an automorphism. The inner reason for this is the fact that one is able to construct points with the cross ratios $\lambda+\mu$, $\lambda-\mu$, $\lambda\mu$, λ/μ by means of projective constructions if points with the cross ratios λ and μ are given.

We define a new collineation by associating

$$\begin{aligned} P &= (x_0, x_1, \cdots, x_n) \\ \to P^* &= (\phi(x_0), \phi(x_1), \cdots, \phi(x_n)) = (x_0^*, x_1^*, \cdots, x_n^*), \\ \epsilon &= (u^{(0)}, u^{(1)}, \cdots, u^{(n)}) \\ \to \epsilon_* &= (\phi(u^{(0)}), \phi(u^{(1)}), \cdots, \phi(u^{(n)})) \\ &= (u^{*(0)}, u^{*(1)}, \cdots, u^{*(n)}). \end{aligned} \tag{6}$$

This is a special collineation (2), the matrix A being the unit matrix. Since the cross ratio is a rational function of the coordinates, points with the cross ratio λ are transformed into points with the cross ratio $\phi(\lambda)$.

If the original collineation is given by (1a), (1b), the transformation $P^* \to P'$, $\epsilon^* \to \epsilon'$ defines a collineation (P and hence P^* ranges over all points, ϵ and hence ϵ^* over all hyperplanes). For this collineation, the cross ratio of four points is neces-

sarily invariant. We see in the usual manner,† that a collineation with this particular property is given by a linear transformation

$$(7) \qquad x_i' = \sum_{k=0}^{n} a_{ik}x_k^*, \qquad u'^{(i)} = \sum_{k=0}^{n} \alpha_{ik}u^{*(k)},$$

where (a_{ik}) is a non-singular matrix and (α_{ik}) its contragredient matrix. On comparing (6) and (7), we obtain (2).

If the only automorphism of the field F is the identity as in the case of the field of the real numbers, then, of course, every collineation is given by a linear transformation $x_i' = \sum a_{ik}x_k$, $u'^{(i)} = \sum \alpha_{ik}u_k$.

4. *Correlation.* We speak of a *correlation* in a projective space whose number of dimensions is larger than 1, if to every point P corresponds a hyperplane $\tilde{\pi}$ in a one-to-one manner and to every hyperplane ρ corresponds a point $\tilde{R}$ in a one-to-one manner, such that $\tilde{\pi}$ passes through $\tilde{R}$ if and only if P lies in ρ.

We define a special correlation by associating the hyperplane $\tilde{u}^{(i)}$ to the point x_i, where

$$(8a) \qquad \tilde{u}^{(i)} = x_i, \qquad (i = 0, 1, 2, \cdots, n),$$

and by associating the point $\tilde{x}_i$ to the hyperplane $u^{(i)}$, where

$$(8b) \qquad \tilde{x}_i = u^{(i)}, \qquad (i = 0, 1, 2, \cdots, n).$$

The inverse of a correlation is a correlation again, the product of two correlations is a collineation. Hence the most general correlation is obtained by performing first a collineation and afterwards the special correlation (8a), (8b). Therefore the most general correlation is given by

† We determine the transformation (7) such that the $n+2$ points $P_0^* = (1, 0, \cdots, 0)$; $P_1^* = (0, 1, 0, \cdots, 0)$; $\cdots$; $P_n^* = (0, \cdots, 0, 1)$; $P_{n+1}^* = (1, 1, \cdots, 1)$ have the same images as they have in the case of the collineation $P^* \rightarrow P'$. Equation (7) sets up a collineation for which the cross ratio is invariant. If we perform the inverse transformation to (7) after the collineation $P^* \rightarrow P'$, we obtain a collineation for which the cross ratio is invariant and the $n+2$ special points P_ρ^* are fixed. Then, however, every point Q is fixed, because the position of Q with respect to $P_0^*, P_1^*, \cdots, P_{n+1}^*$ can be described uniquely by means of cross ratios only (projective coordinates). Therefore, the collineation (7) and the collineation $P^* \rightarrow P'$ map all points exactly in the same manner, and both are identical.

$$(9) \qquad \tilde{u}^{(i)} = \sum_{k=0}^{n} a_{ik}\phi(x_k), \qquad \tilde{x}_i = \sum_{k=0}^{n} \alpha_{ik}\phi(u^{(k)}),$$

where again (a_{ik}) and (α_{ik}) are non-singular contragredient matrices and $\lambda \to \phi(\lambda)$ is an automorphism of the underlying field F.

5. *Correlations Defining a Null System.*

THEOREM. *If a correlation associates to every point P a hyperplane $\tilde{\pi}$ passing through P, the number n of dimensions of the space is odd and the correspondence defines a null system. The equation of $\tilde{\pi}$ in point coordinates is given by the vanishing of a skew-symmetric bilinear form, one set of variables being the coordinates of P, the other one the coordinates of a given variable point of $\tilde{\pi}$.*

PROOF. If the correlation is given by (9), then

$$\sum_{i,k=0}^{n} a_{ik}x_i\phi(x_k) = 0$$

must hold for every point P since x_i lies on $\tilde{u}^{(i)}$. Every automorphism of a field F leaves 0 and 1 unaltered. We put $x_i = 1$, $x_j = 0$ for $j \neq i$ and obtain $a_{ii} = 0$ for all i. Hereafter we set $x_i = 1$ and leave x_k indeterminate for the moment for a fixed pair $i \neq k$. We set $x_j = 0$ for $j \neq i, k$. We get

$$(10) \qquad a_{ik}\phi(x_k) + a_{ki}x_k = 0.$$

If we put $x_k = 1$, we see that $a_{ik} = -a_{ki}$, the matrix (a_{ik}) is skew-symmetric. Furthermore, there exists a pair i, k for which $a_{ik} \neq 0$. By using the result we have just found we derive from (10), $\phi(x_k) = x_k$ for all elements x_k in F. Hence $\lambda \to \phi(\lambda)$ is the identical automorphism.

Then the condition for a point $(y_0, y_1, \cdots, y_n)$ to lie in the hyperplane $\tilde{u}^{(i)}$ corresponding to the point x_i is given according to (9) by $\sum a_{ik}y_ix_k = 0$, where (a_{ik}) is skew-symmetric as required by the theorem. The number of dimensions n is odd because otherwise the determinant of (a_{ik}) would vanish.

A one-to-one correspondence between points and hyperplanes with the properties (1) and (2) defined in the introduction establishes a correlation of period 2. It follows that we have a null system.

6. *General Correlations of Period* 2. We consider more general correlations of period 2 and prove the following theorem.

THEOREM. *A correlation of period* 2 *associates to a point* $(x_0, \cdots, x_n)$ *a hyperplane whose equation in point coordinates* z_i *is given either by*

$$\sum_{i,k} a_{ik}z_ix_k = 0 \text{ with } a_{ik} = a_{ki}, \quad (\textit{polarity}),$$

or by

$$\sum_{i,k} a_{ik}z_ix_k = 0 \text{ with } a_{ik} = -a_{ki}, \quad (\textit{null system for an odd number of dimensions only}),$$

or by the vanishing of a Hermitian form†

$$\sum a_{ik}z_i\phi(x_k) = 0, \qquad a_{ik} = \phi(a_{ki}),$$

where $\lambda \to \phi(\lambda)$ *is an automorphism of period* 2 *of the underlying field* F.

PROOF. If a correlation of period 2 is given by (9), we derive, by effecting the correlation twice,

$$\gamma x_i = \sum_{j=0}^{n} \alpha_{ij}\phi(\bar{u}^{(j)}) = \sum_{j,k=0}^{n} \alpha_{ij}\phi(a_{jk})\phi\phi(x_k),$$

where $\gamma \neq 0$ is a factor depending on $(x_0, x_1, \cdots, x_n)$ but independent of i.

Besides the matrix $A = (a_{ik})$ we consider $A^* = (\phi(a_{ik}))$, and form

$$M = A'^{-1}A^*. \tag{11}$$

Then, by the linear transformation with this matrix,

$$(\phi\phi(x_0), \phi\phi(x_1), \cdots, \phi\phi(x_n))$$

is transformed into

$$(\gamma x_0, \gamma x_1, \cdots, \gamma x_n).$$

† In the general sense in which this word is used by L. E. Dickson in *Modern Algebraic Theories*, p. 66. The first case can, of course, be considered as a special case of the third case.

By putting all $x_k = 0$ except one which has the value 1, we prove in the usual manner that M is a diagonal matrix. By putting all $x_i = 1$, we see that $M = c \cdot 1$, where 1 denotes the unit matrix and $c \neq 0$. Now we set $x_1 = 1$, $x_2 = \lambda$, $x_3 = x_4 = \cdots = 0$, where λ is an arbitrary element of F. Then we find

$$\gamma = c \cdot 1, \qquad \gamma \cdot \lambda = c \cdot \phi\phi(\lambda),$$

and hence $\phi\phi(\lambda) = \lambda$; the automorphism $\lambda \to \phi(\lambda)$ is of period 2. Equation (11) in combination with $M = c \cdot 1$ gives

$$A^* = cA'. \tag{12}$$

We apply the automorphism $\lambda \to \phi(\lambda)$ to the equation (12). Then $A^* = (\phi(a_{ik}))$ is transformed into $(\phi\phi(a_{ik})) = (a_{ik}) = A$ and hence

$$A = \phi(c)A^{*\prime}. \tag{13}$$

From (12) and (13) follows $A = \phi(c) \cdot c \cdot A$, and, since $A \neq 0$,

$$\phi(c) \cdot c = 1. \tag{14}$$

If $c \neq -1$, we set $l = 1 + c$ and obtain

$$c \cdot \phi(l) = c(1 + \phi(c)) = c + c\phi(c) = c + 1 = l,$$

so that

$$c = l/(\phi(l). \tag{15}$$

We then replace A by lA, as we may, and consequently A^* by $\phi(l)A^*$. Then in (12), c is to be replaced by $c\phi(l)/l = 1$. Therefore we can assume $c = 1$, and (12) shows that A is Hermitian with respect to the automorphism $\lambda \to \phi(\lambda)$ of period 2.

If $c = -1$, but if there exist elements μ in F with $\phi(\mu) \neq \mu$, (15) holds for $l = \mu - \phi(\mu)$ and we obtain the same result.

If, finally, $\phi(\mu) = \mu$ for all μ in F, we have $A^* = A$ and from (14) follows $c^2 = 1$, $c = \pm 1$, $A' = \pm A$. Therefore, A is either symmetric or skew-symmetric.

If $(z_0, z_1, \cdots, z_n)$ is a point of the hyperplane corresponding to $(x_0, x_1, \cdots, x_n)$, it follows from (9) that $\sum a_{ik} z_i \phi(x_k) = 0$. We have shown that here ϕ is of period 2 and (a_{ik}) Hermitian, or $\phi(\lambda) = \lambda$ identically and (a_{ik}) symmetric or skew-symmetric.

THE INSTITUTE FOR ADVANCED STUDY

COMPLEXES AND MANIFOLDS REPRESENTED BY FUNCTIONS OF REAL VARIABLES*

BY W. M. WHYBURN

It is the purpose of this paper to show that a wide class of loci in a real space of n dimensions is composed of sets which are complexes and manifolds in the sense of combinatorial analysis situs. The combinatorial approach to analysis raises the important question of determining the conditions that a real function $u=f(x_1, \cdots, x_n)$ of a finite set of real variables must satisfy in order that this function generate a locus which can be identified as a cell, complex, or manifold.† Answers to this question form essential links between combinatorial analysis situs and real variable theory. Van der Waerden,‡ Lefschetz,§ Koopman and Brown,‖ have studied this question for cases where the function u is analytic. S. S. Cairns¶ has investigated the cases that arise for functions u that are continuous together with their first partial derivatives. The present paper deals with functions u which are continuous but which are not required to have first partial derivatives at any points of their domains of definition. The restrictions placed on the functions are essentially those used by Hedrick and Westfall** in their general

* Presented to the Society, November 30, 1935.

† See Veblen, *Analysis Situs*, Colloquium Publications of this Society, vol. 5, part 2.

‡ Mathematische Annalen, vol. 102 (1929), pp. 360–361.

§ *Topology*, Colloquium Publications of this Society, vol. 12, chapter 8. On page 364, Lefschetz expressed a belief that van der Waerden (loc. cit.) was first to establish connections between functionally defined loci and complexes. It should be mentioned that a slightly earlier paper by W. M. Whyburn [this Bulletin, vol. 35 (1929), p. 706] contained a proof that a locus of a general type was a manifold and also carried a reference to results of this type which S. S. Cairns had obtained but had not published.

‖ Transactions of this Society, vol. 34 (1932), pp. 231–251.

¶ See his papers in Annals of Mathematics, (2), vol. 35 (1934), pp. 579–587, and this Bulletin, vol. 41 (1935), pp. 549–552, where references to his earlier work will be found.

** Bulletin de la Société Mathématique, vol. 44, pp. 1–14, and Festschrift David Hilbert, Berlin, 1922, pp. 74–77, or Mathematische Annalen, vol. 85 (1922), pp. 74–77.

implicit function theorem. The use of this latter theorem makes it possible for proofs to be given without the use of partial derivatives, directional derivatives, or direction cosines. This omission of derivatives is desirable since derivatives do not have a role in analysis situs. Continuity, on the other hand, enters into the transformations and correspondences of analysis situs.

THEOREM 1. *Let* $H: A_i \leqq x_i \leqq B_i$, $(i=1, 2, \cdots, n)$, *be the interior and boundary of an n-rectangle in the space of the n real variables* $x_1, x_2, \cdots, x_n$, *and let K be a closed connected subset of H. For each point P of K, let there exist an n-rectangular neighborhood of P,* $H_P: A_{Pi} \leqq x_i \leqq B_{Pi}$, $(i=1, \cdots, n)$, *such that the subset of K that belongs to* H_P *is identical with the set of points which satisfies a single equation* $x_j = f(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)$, *where f is continuous and* $A_{Pj} < f < B_{Pj}$ *on the set of points* $A_{Pi} \leqq x_i \leqq B_{Pi}$, $(i=1, 2, \cdots, j-1, j+1, \cdots, n)$. *Under these hypotheses, the point set K is an* $(n-1)$*-complex.**

PROOF. Let $[H_P]$ be the collection of rectangles obtained when each point P of K is covered as indicated in the hypotheses of the theorem. Since K is closed and bounded, the Heine-Borel theorem may be applied to get a finite sub-collection of this collection which covers K. Let such a finite sub-collection be chosen and let $H_1, H_2, \cdots, H_m$ be the finite collection of n-polygons formed from this collection when each pair of rectangles of the collection that have interior points in common, and that have the further property that the subsets of K belonging to them are represented by functions of the same $n-1$ variables, is replaced by their sum and this process is continued until no two of the sets $H_1, \cdots, H_m$ which have interior points in common have the same $n-1$ variables of the set $(x_1, \cdots, x_n)$ as independent variables in the functions that represent the subsets of K that they contain. Let H_i' be the boundary of H_i and p be a point of K that belongs to $H-(H'+H_1'+\cdots+H_m')$. Let h_p be the maximal connected subset of $H-(H'+H_1'+\cdots+H_m')$ that contains p, together with the boundary of this maximal connected subset. If $[h_p]$ denotes the collection of sets obtained in this way, then $[h_p]$ consists of a finite set, $h_1, h_2, \cdots, h_q$, of

* The terms "complex," "cell," "circuit," "manifold," $\cdots$, are used with the meanings given to them in Veblen's book, loc. cit.

n-polygons (interiors and boundaries) each of which has its face $(n-1)$-polygons subsets of the faces of $H, H_1, \cdots, H_m$.

LEMMA 1. *Every point of K is a limit point of a subset of K that is interior to some one of the polygons $h_1, h_2, \cdots, h_q$.*

PROOF OF LEMMA 1. The connectivity of K shows that any point of K which is interior to one of the sets $h_1, \cdots, h_q$, is a limit point of points of K interior to this. If $p:(a_1, a_2, \cdots, a_n)$ is a point of K not interior to any h_j, p is interior to some H_r of the set $H_1, \cdots, H_m$, and, with a suitable permutation of the subscripts of the x's, the points of K that belong to H_r are solutions of the equation $x_1 = f(x_2, \cdots, x_n)$. We may therefore pick a sequence of points $p_i:(a_{i1}, \cdots, a_{in})$, $(i=1, 2, \cdots)$, of K with p as its sequential limit point and such that $a_{i2} \neq a_2$, $a_{i3} \neq a_3, \cdots, a_{in} \neq a_n$, $(i=1, 2, \cdots)$. If $x = a_1$ is not a face that contains p and belongs to any polygon of the set $h_1, \cdots, h_q$, then infinitely many of the points p_i are interior to some one of the finite set $h_1, \cdots, h_q$ and p is on the boundary of this set. If $x = a_1$ is a face that contains p and belongs to one of the h's, then this face belongs to one of the polygons H_r, $s \neq r$, in which an equation $x_2 = g(x_1, x_2, \cdots, x_n)$ holds (after a suitable choice of subscripts on the x's). If we make use of the single-valuedness and continuity of f and g, we are able to choose the points p_i so that they have the additional property $a_{i1} \neq a_1$, $(i=1, 2, \cdots)$. With this choice of the points $p_1, p_2, \cdots$, only a finite number of them will lie on faces of the polygons $h_1, \cdots, h_q$ and hence infinitely many of them will be interior to some one of this finite set of polygons and p will be on the boundary of such a polygon. This completes the proof of the lemma.

LEMMA 2. *Let K_i be the subset of K that is on the boundary of h_i and is such that each point of K_i is a limit point of points of K that are interior to h_i. The set K_i is a finite collection of $(n-2)$-circuits each of which is the boundary of an $(n-1)$-cell that is a subset of K and is interior to h_i.*

PROOF OF LEMMA 2. We use mathematical induction. The case $n=2$ yields a set of plane polygons for $h_1, h_2, \cdots, h_q$, with faces parallel to the coordinate axes. Let h_i be a subset of H_r and let $x_2 = f(x_1)$ be the equation associated with H_r. Since this equation is satisfied by the subset of K in h_i, it follows that K_i

has at most one point, $[c, f(c)]$, in any face $x_1 = c$ of h_i. If h_i has a face $x_2 = c$ which contains a point of K_i, then such a face must belong to a polygon of the set $H_1, \cdots, H_m$ in which an equation $x_1 = g(x_2)$ holds, since H_r has no face of the form $x_2 = constant$ which contains a point of K. The equation $x_1 = g(x_2)$ yields at most one point of K_i on this face. The set K_i consists of a finite number of zero cells and the single-valuedness of f and g shows that each of these can be on the boundary of at most two maximal connected subsets of K that belong to the interior of h_i. If k is a maximal connected subset of K that is interior to h_i, the equation $x_2 = f(x_1)$ establishes a one-to-one continuous correspondence between k_1 and a closed interval on the x_1 axis, where k_1 is k together with its boundary. The set k_1 is therefore an $(n-1)$-complex consisting of an $(n-1)$-cell interior to h_i and its $(n-2)$-circuit boundary composed of two of the zero cells of K_i. Since each point of K_i is on the boundary of at least one set of type k and is not on the boundary of more than two such sets, it follows that the lemma is true for the case $n = 2$.

Let the lemma hold up to and including the case $n-1$. In the case of n variables, let h_i be one of the polygons of the set $h_1, \cdots, h_q$ and let H_r be a polygon of the set $H_1, \cdots, H_m$ that contains h_i. Let the equation $x_1 = f(x_2, \cdots, x_n)$ hold for the subset of K that belongs to H_r. The subset of K_i that belongs to any face of the form $x_j = c$, $(j = 2, \cdots, n)$, satisfies the equation $x_1 = f(x_2, \cdots, x_{j-1}, c, x_{j+1}, \cdots, x_n) = F(x_2, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)$. The single-valuedness of F together with an application of the lemma for the case $n-1$ shows that this intersection consists of a finite number of $(n-2)$-cells and their bounding $(n-3)$-circuits. If K_i intersects a face of the type $x_1 = c$, where c is a constant, then such an intersection satisfies an equation of the form $x_2 = g(x_1, x_3, \cdots, x_n)$ (since such a face must belong to some H_j other than H_r). The single-valuedness of g together with an application of the lemma for $n-1$ shows that this intersection is a finite set of $(n-2)$-cells together with their bounding $(n-2)$-circuits. Let k be a maximal connected subset of K that is interior to h_i and let k_1 denote k together with its boundary. The equation $x_1 = f(x_2, \cdots, x_n)$ establishes a one-to-one continuous correspondence between k_1 and an $(n-1)$-complex [composed of an $(n-2)$-circuit and the $(n-1)$-cell that it bounds] in the hyperplane $x_1 = 0$. Hence k_1 is an $(n-1)$-

complex consisting of an $(n-1)$-cell and its bounding $(n-2)$-circuit. The lemma follows at once when it is noted that each point of K_i is on the boundary of at least one maximal connected subset of type k and (from the single-valuedness of the functions f and g) no such point is on the boundary of more than two sets of type k.

PROOF OF THEOREM 1. The theorem follows at once from the lemmas. The set K is an $(n-1)$-complex composed of the $(n-1)$-cells of Lemma 2 together with their boundaries which are the $(n-2)$-circuits described in that lemma. Lemma 1 shows that all of K belongs to this complex, while the fact that the $(n-1)$-cells are mutually exclusive follows from the observation that no two of the sets $h_1, \cdots, h_q$ have interior points in common. A subdivision of the $(n-2)$-circuits of this complex into cells of orders $n-2, n-3, \cdots, 0$ is obtained when the reasoning used in the proof of Lemma 2 is repeated, with slight verbal modifications, for the $(n-1)$-polygons obtained on the faces of $h_1, \cdots, h_q$.

THEOREM 2. *Under the hypotheses of Theorem* 1, *if the set* K *is interior to* H, K *is a closed* $(n-1)$*-manifold.*

PROOF. From Theorem 1, K is an $(n-1)$-complex. Since K is interior to H, we may make the neighborhood H_P of Theorem 1 so that it is interior to H. The equation $x_j=(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)$ establishes a one-to-one continuous correspondence between the subset of K in H_P and the interior and boundary of an $(n-1)$-rectangle in the hyperplane $x_j=0$. Hence the subset of K in H_P is an $(n-1)$-cell and boundary and thus each point of K has a neighborhood which is an $(n-1)$-cell. Hence K is a closed $(n-1)$-manifold.*

COROLLARY. *If* K *is interior to* H *and the hypotheses of Theorem* 1 *are satisfied,* K *is an* $(n-1)$*-circuit that forms the boundary of an* n*-cell interior to* H.

THEOREM 3. *Let the real function* $f(x_1, \cdots, x_n)$ *be contin-*

* This theorem would also follow as a corollary to the proof of Theorem 1 if it were observed that this proof shows each $(n-2)$-cell of the complex K to be on the boundary of at least one but not more than two $(n-1)$-cells. When K is interior to H, the single-valued functions that define K in a neighborhood of any point p make each $(n-2)$-cell incident with exactly two $(n-1)$-cells.

uous in an open portion D of the space of the n real variables $(x_1, \cdots, x_n)$. Let H be an n-rectangle together with its interior which is a subset of D and let K be the points of H which satisfy $f(x_1, \cdots, x_n) = 0$. For each point p of K, let there exist an n-rectangular neighborhood throughout which some one of the difference quotients a_j, $(j = 1, 2, \cdots, n)$, is definite in sign, where

$$a_j = \frac{f(x_1, \cdots, x_{j-1}, x_j', x_{j+1}, \cdots, x_n) - f(x_1, \cdots, x_n)}{x_j' - x_j}.$$

Under these hypotheses, the set K is an $(n-1)$-complex. Furthermore, if K is interior to H, K is a finite set of closed $(n-1)$-manifolds.

PROOF. This theorem is an immediate consequence of Theorems 1 and 2 when account is taken of an implicit function theorem by Hedrick and Westfall.* This implicit function theorem applies directly to show that the hypotheses of Theorem 1 are satisfied for each connected subset of K. The same implicit function theorem shows that K consists of a finite number of maximal connected subsets. The latter fact follows from the observation that the sum of infinitely many such subsets would have a limit point in K and the uniqueness part of the implicit function theorem would break down at such a limit point. The conclusions of Theorem 3 follow from the conclusions of Theorems 1 and 2.

THEOREM 4. *Let $H: A_i \leqq x_i \leqq B_i$, $(i = 1, 2, \cdots, n)$, be the interior and boundary of an n-rectangle in the space of the n real variables $(x_1, \cdots, x_n)$ and let K be a closed, connected subset of H. For each point p of H, let there exist an n-rectangular neighborhood $H_p: A_{pi} \leqq x_i \leqq B_{pi}$, $(i = 1, \cdots, n)$, of p such that the subset of K that belongs to H_p is identical with the set of points which satisfies a unique system of r equations:*

$$x_{j_1} = g_1(x_{j_{r+1}}, \cdots, x_{j_n}), \qquad x_{j_2} = g_2(x_{j_{r+1}}, \cdots, x_{j_n}),$$
$$\cdots, \qquad x_{j_r} = g_r(x_{j_{r+1}}, \cdots, x_{j_n}),$$

where $j_1, \cdots, j_n$ is an arrangement of the indices $1, 2, \cdots, n$ that is fixed for each H_p, and where the functions $g_1, \cdots, g_r$ are

* Loc. cit.

continuous and satisfy $A_{pi_1} < g_1 < B_{pi_1}, \cdots, A_{pi_r} < g_r < B_{pi_r}$ *on the point set* $A_{pi} \leqq x_i \leqq B_{pi}$, $(i = j_{r+1}, \cdots, j_n)$. *Under these hypotheses, the point set K is an* $(n-r)$*-complex.*

PROOF. Let n-polygons $h_1, \cdots, h_q$ be constructed as they were in the proof of Theorem 1 (with the variation that the role of the single function f of Theorem 1 is now played by the r functions $g_1, \cdots, g_n$). Consider the subset k of K that belongs to h_i and let the notation be chosen so that $x_1 = g_1(x_{r+1}, \cdots, x_n), \cdots, x_r = g_r(x_{r+1}, \cdots, x_n)$ are the equations satisfied by the points of k. The point set k is in one-to-one continuous *correspondence with a set of* points k' in the $n-r+1$ plane: $x_2 = 0, \cdots, x_r = 0$ if we let the point $P(x_1, x_2, \cdots, x_r, x_{r+1}, \cdots, x_n)$ of k correspond to the point $P'(x_1, 0, 0, \cdots, 0, x_{r+1}, \cdots, x_n)$ of k'. Furthermore, the set k' satisfies the equation $x_1 = g_1(x_{r+1}, \cdots, x_n)$. An application of Theorem 1 for the case $n-r+1$ shows that k' is an $(n-r)$-complex. Since k is in one-to-one continuous correspondence with k', it follows that k is also an $(n-r)$-complex. An application of this same type of reasoning shows that the subset of k that belongs to any face of h_i as well as the subset of k that is common to a face of h_i and a face of h_j, $j \neq i$, is an $(n-r-1)$-complex. It follows that K is an $(n-r)$-complex with its $(n-r)$-cells interior to the polygons $h_1, \cdots, h_q$ and that the $(n-r-1)$-circuits that bound these cells lie in the boundaries of these polygons.

An adaptation of the proof of Theorem 2 yields the following theorem.

THEOREM 5. *Under the hypotheses of Theorem* 4, *if the set K is interior to the rectangle H, then K is a closed* $(n-r)$*-manifold.*

THEOREM 6. *Let the real functions* $f_1(x_1, \cdots, x_n), \cdots, f_r(x_1, \cdots, x_n)$ *be continuous in an open portion D of the space of the n real variables* $(x_1, \cdots, x_n)$. *Let* $H: A_i \leqq x_i \leqq B_i$, $(i = 1, \cdots, n)$, *be an n-rectangular region which is a subset of D and let K be the set of points of H which satisfy the equations* $f_1(x_1, \cdots, x_n) = 0, \cdots, f_r(x_1, \cdots, x_n) = 0$. *For each point p of K let there exist an n-rectangular neighborhood of p and an arrangement* $j_1, \cdots, j_n$ *of the indices* $1, 2, \cdots, n$ *such that if* $y_i = x_{j_i}$, $(i = 1, \cdots, n)$, $g_m(y_1, \cdots, y_n) = f_m(x_1, \cdots, x_n)$, $(m = 1, 2, \cdots, r)$, *then the difference Jacobian*

$$J = \begin{vmatrix} a_{11} & a_{12} \cdots a_{1r} \\ a_{21} & a_{22} \cdots a_{2r} \\ \cdot & \cdot \cdot \cdot \cdot \cdot \cdot \\ a_{r1} & a_{r2} \cdots a_{rr} \end{vmatrix},$$

$$a_{ik} = \frac{[g_i(y_1, \cdots, y_{k-1}, y_k', y_{k+1}^0, \cdots, y_r^0, y_{r+1}, \cdots, y_n) - g_i(y_1, \cdots, y_k, y_{k+1}^0, \cdots, y_r^0, y_{r+1}, \cdots, y_n)]}{[y_k' - y_k]},$$

as well as the elements of the principal diagonal and the minor obtained by deleting the first i rows and the first i columns, $(i = 1, 2, \cdots, r-1)$, *are definite for all choices of the points in this neighborhood. Under these hypotheses, the set* K *is an* $(n-r)$*-complex. Furthermore, if* K *is interior to* H, *it is a finite set of closed* $(n-r)$*-manifolds.*

PROOF. An application of the Hedrick-Westfall implicit function theorem (loc. cit.) shows that K consists of a finite collection $K_1, \cdots, K_m$ of maximal connected subsets each of which satisfies the hypotheses of Theorem 4. An application of Theorem 4 to each of these sets yields the fact that K is an $(n-r)$-complex. In the case where K is interior to H, Theorem 5 shows that $K_1, \cdots, K_m$ are closed $(n-r)$-manifolds.

In conclusion, it might be remarked that the hypotheses of Theorems 3 and 6 do not require the existence of derivatives of the functions that enter into these theorems.* On the other hand, the hypotheses of these theorems are met if the ordinary implicit function theorem hypotheses (involving derivatives of the first order) are valid. This omission of derivatives seems desirable since properties of analysis situs are not dependent upon derivatives. The assumption of continuity seems more essential since the transformations and correspondences of analysis situs are required to be continuous.

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* An interesting example of a function that meets the hypotheses of these theorems and of the Hedrick-Westfall theorem but not those of the ordinary implicit function theorem, is obtained by making use of the well known example of a continuous, monotone, non-absolutely continuous function (see Hille and Tamarkin, American Mathematical Monthly, vol. 36 (1929), pp. 255–264). If x is added to this function, an increasing function is obtained. The Hedrick-Westfall theorem applies directly to yield an inverse function $x = g(y)$.

REDUCIBLE BOOLEAN FUNCTIONS

BY J. C. C. McKINSEY

In this note I establish a condition that a Boolean function of n variables, say f, be reducible to a product of two Boolean functions f_1 and f_2, where f involves variables not occurring in f_i; and, similarly, that f be reducible to f_1+f_2, or to $f_1 \circ f_2$, or to $f_1 \Delta f_2$.* These results are of interest in connection with the general theory of Boolean operations, since every Boolean operation can be regarded as a Boolean function.

In order to state my results briefly, I use the symbol $\oplus$, which stands ambiguously for any one of the four operations $\times$, $+$, $\circ$, Δ. Thus each of my theorems really comprises four theorems, which can be obtained from the given theorem by substituting first $\times$ for $\oplus$ throughout, then $+$, then $\circ$, and then Δ. The theorems now follow.

THEOREM 1. *If a Boolean function*

$$f(x_1, \cdots, x_n)$$

be given, then a necessary and sufficient condition that there exist a g and an h, so that

$$f(x_1, \cdots, x_n) = g(x_1, \cdots, x_p) \oplus h(x_q, \cdots, x_n), \qquad (q \leqq p+1),$$

is that

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* The operation $a \circ b$ is defined by $a \circ b = ab' + a'b$; and the operation $a \Delta b$ is defined by $a \Delta b = ab + a'b'$. These operations, which are mutually dual, are associative and commutative, and satisfy the further laws: $(a \circ b)' = a \circ b' = a \Delta b$, $a \circ 1 = a \Delta 0 = a'$, $a \circ a = a \Delta a' = 0$, $a \circ 0 = a \Delta 1 = a$. For a further discussion, see Bernstein's paper, *Postulates for Boolean algebra involving the operation of complete disjunction*, to appear in the Annals of Mathematics. For a detailed treatment of $a \circ b$, see the two papers by M. H. Stone: *Postulates for Boolean algebra and generalized Boolean algebra*, American Journal of Mathematics, vol. 57 (1935), pp. 703–732, and *Subsumption of the theory of Boolean algebras under the theory of rings*, Proceedings of the National Academy of Sciences, vol. 21 (1935), pp. 103–105. Stone writes $a \Delta b$ for what I designate by $a \circ b$, and he does not discuss the relation which I denote by $a \Delta b$.

$$f(x_1, \cdots, x_{q-1}, z_q, \cdots, z_p, x_{p+1}, \cdots, x_n)$$
$$\oplus f(y_1, \cdots, y_{q-1}, z_q, \cdots, z_p, y_{p+1}, \cdots, y_n)$$
$$= f(x_1, \cdots, x_{q-1}, z_q, \cdots, z_p, y_{p+1}, \cdots, y_n)$$
$$\oplus f(y_1, \cdots, y_{q-1}, z_q, \cdots, z_p, x_{p+1}, \cdots, x_n)$$

*identically.**

PROOF. I first prove the theorem for the case that $q=p+1$. To see that the condition is necessary (when $q=p+1$), suppose that there exists a g and an h so that

(1) $$f(x_1, \cdots, x_n) = g(x_1, \cdots, x_p) \oplus h(x_{p+1}, \cdots, x_n).$$

Then

(2)
$$\begin{aligned} f(x_1, \cdots, x_n) &\oplus f(y_1, \cdots, y_n) \\ &= [g(x_1, \cdots, x_p) \oplus h(x_{p+1}, \cdots, x_n)] \\ &\quad \oplus [g(y_1, \cdots, y_p) \oplus h(y_{p+1}, \cdots, y_n)] \\ &= [g(x_1, \cdots, x_p) \oplus h(y_{p+1}, \cdots, y_n)] \\ &\quad \oplus [g(y_1, \cdots, y_p) \oplus h(x_{p+1}, \cdots, x_n)] \\ &= f(x_1, \cdots, x_p, y_{p+1}, \cdots, y_n) \\ &\quad \oplus f(y_1, \cdots, y_p, x_{p+1}, \cdots, x_n), \end{aligned}$$

as was to be shown.

To show that the condition is sufficient (when $q=p+1$), suppose that $f(x_1, \cdots, x_n)$ is such that (2) holds. We must now distinguish the four cases. If $\oplus = \times$, define

$$g(x_1, \cdots, x_p) = \sum_{z_i=0,1} f(x_1, \cdots, x_p, z_{p+1}, \cdots, z_n),$$
$$h(x_{p+1}, \cdots, x_n) = \sum_{z_i=0,1} f(z_1, \cdots, z_p, x_{p+1}, \cdots, x_n).$$

Then

$$g(x_1, \cdots, x_p)h(x_{p+1}, \cdots, x_n)$$
$$= \left[\sum_{z_i=0,1} f(x_1, \cdots, x_p, z_{p+1} \cdots, z_n) \right]$$

* Or, what amounts to the same thing, that the stated condition hold for the x's, y's, z's=0, 1. Thus whether the condition is satisfied can be seen directly from the discriminants of f.

$$\cdot \left[\sum_{z_i=0,1} f(z_1, \cdots, z_p, x_{p+1}, \cdots, z_n) \right]$$

$$= \sum_{z_i=0,1} f(x_1, \cdots, x_p, z_{p+1}, \cdots, z_n) f(z_1, \cdots, z_p, x_{p+1}, \cdots, x_n)$$

$$= \sum_{z_i=0,1} f(x_1, \cdots, x_n) f(z_1, \cdots, z_n)$$

$$= f(x_1, \cdots, x_n) \sum_{z_i=0,1} f(z_1, \cdots, z_n)$$

$$= f(x_1, \cdots, x_n).$$

If $\oplus = +$, define

$$g(x_1, \cdots, x_p) = \prod_{z_i=0,1} f(x_1, \cdots, x_p, z_{p+1}, \cdots, z_n),$$

$$h(x_{p+1}, \cdots, x_n) = \prod_{z_i=0,1} f(z_1, \cdots, z_p, x_{p+1}, \cdots, x_n),$$

and proceed as above. If $\oplus = o$, define

$$g(x_1, \cdots, x_p) = f(0, \cdots, 0) \circ f(x_1, \cdots, x_p, 0, \cdots, 0),$$

$$h(x_{p+1}, \cdots, x_n) = f(0, \cdots, 0, x_{p+1}, \cdots, x_n).$$

Then

$$\begin{aligned} &g(x_1, \cdots, x_p) \circ h(x_{p+1}, \cdots, x_n) \\ &\quad = f(0, \cdots, 0) \circ f(x_1, \cdots, x_p, 0, \cdots, 0) \\ &\qquad \circ f(0, \cdots, 0, x_{p+1}, \cdots, x_n) \\ &\quad = f(0, \cdots, 0) \circ f(0, \cdots, 0) \circ f(x_1, \cdots, x_n) \\ &\quad = 0 \circ f(x_1, \cdots, x_n) \\ &\quad = f(x_1, \cdots, x_n). \end{aligned}$$

If $\oplus = \Delta$, define

$$g(x_1, \cdots, x_p) = f(0, \cdots, 0) \Delta f(x_1, \cdots, x_p, 0, \cdots, 0),$$

$$h(x_{p+1}, \cdots, x_n) = f(0, \cdots, 0, x_{p+1}, \cdots, x_n),$$

and proceed as above. This completes the proof of the theorem for the case $q = p+1$.

For the general case, to say that

$$(3) \qquad f(x_1, \cdots, x_n) = g(x_1, \cdots, x_p) \oplus h(x_q, \cdots, x_n)$$

holds for every $x_q, \cdots, x_p$ is equivalent to saying it holds for $x_q, \cdots, x_p = 0, 1$. Hence saying that (3) holds for every $x_1, \cdots, x_n$ is equivalent to saying that the following equations hold for every $x_1, \cdots, x_{q-1}, x_{p+1}, \cdots, x_n$:

$$
\begin{aligned}
&f(x_1, \cdots, x_{q-1}, 1, \cdots, 1, x_{p+1}, \cdots, x_n) \\
&\quad = g(x_1, \cdots, x_{q-1}, 1, \cdots, 1) \oplus h(1, \cdots, 1, x_{p+1}, \cdots, x_n), \\
&\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad (4) \\
&f(x_1, \cdots, x_{q-1}, 0, \cdots, 0, x_{p+1}, \cdots, x_n) \\
&\quad = g(x_1, \cdots, x_{q-1}, 0, \cdots, 0) \oplus h(0, \cdots, 0, x_{p+1}, \cdots, x_n).
\end{aligned}
$$

It will be noticed that each discriminant of $g(x_1, \cdots, x_p)$, and each discriminant of $h(x_q, \cdots, x_n)$, occurs in *just one* of the equations (4). Hence to say that there exists a $g(x_1, \cdots, x_p)$ and an $h(x_q, \cdots, x_n)$ so that (3) holds identically is equivalent to saying that there exist elements $g(1, \cdots, 1, 1, \cdots, 1), \cdots, g(0, \cdots, 0, 1, \cdots, 1), h(1, \cdots, 1, 1, \cdots, 1), \cdots, h(1, \cdots, 1, 0, \cdots, 0)$ so that the first equation of (4) holds identically, and similarly for each of the other equations of (4). Hence, by the first part of the proof, to say that (3) holds identically is equivalent to saying that the following set of equations hold identically:

$$
\begin{aligned}
&f(x_1, \cdots, x_{q-1}, 1, \cdots, 1, x_{p+1}, \cdots, x_n) \\
&\qquad \oplus f(y_1, \cdots, y_{q-1}, 1, \cdots, 1, y_{p+1}, \cdots, y_n) \\
&\quad = f(x_1, \cdots, x_{q-1}, 1, \cdots, 1, y_{p+1}, \cdots, y_n) \\
&\qquad \oplus f(y_1, \cdots, y_{q-1}, 1, \cdots, 1, x_{p+1}, \cdots, x_n), \\
&\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad (5) \\
&f(x_1, \cdots, x_{q-1}, 0, \cdots, 0, x_{p+1}, \cdots, x_n) \\
&\qquad \oplus f(y_1, \cdots, y_{q-1}, 0, \cdots, 0, y_{p+1}, \cdots, y_n) \\
&\quad = f(x_1, \cdots, x_{q-1}, 0, \cdots, 0, y_{p+1}, \cdots, y_n) \\
&\qquad \oplus f(y_1, \cdots, y_{q-1}, 0, \cdots, 0, x_{p+1}, \cdots, x_n).
\end{aligned}
$$

But the set (5) is equivalent to the single condition given in the statement of the theorem.

I now give a partial generalization of Theorem 1 to the case where $f(x_1, \cdots, x_n)$ is reducible to a product (or sum, $\cdots$) of *three* functions. The generalization is not complete, in that it

covers only the case where no two of the three functions involve a common variable. The proof, which is closely analogous to the first part of the proof of Theorem 1, is omitted.

THEOREM 2. *If a Boolean function* $f(x_1, \cdots, x_n)$ *be given, then a necessary and sufficient condition that there exist functions* g_1, g_2, g_3 *so that*

$$f(x_1, \cdots, x_n) = g_1(x_1, \cdots, x_p) \oplus g_2(x_{p+1}, \cdots, x_q) \oplus g_3(x_{q+1}, \cdots, x_n)$$

is that

$$f(x_1, \cdots, x_n) \oplus f(y_1, \cdots, y_n) = f(x_1, \cdots, x_p, y_{p+1}, \cdots, y_n) \oplus f(y_1, \cdots, y_p, x_{p+1}, \cdots, x_n) = f(x_1, \cdots, x_q, y_{q+1}, \cdots, y_n) \oplus f(y_1, \cdots, y_q, x_{q+1}, \cdots, x_n)$$

identically.

From Theorems 1 and 2 we have immediately the following theorem.

THEOREM 3. *If there exist functions* g_1 *and* g_2 *and also functions* g_3 *and* g_4 *so that*

$$f(x_1, \cdots, x_n) = g_1(x_1, \cdots, x_p) \oplus g_2(x_{p+1}, \cdots, x_n),$$
$$f(x_1, \cdots, x_n) = g_3(x_1, \cdots, x_q) \oplus g_4(x_{q+1}, \cdots, x_n), \quad (p < q),$$

then there exist functions k_1, k_2, k_3 *so that*

$$f(x_1, \cdots, x_n) = k_1(x_1, \cdots, x_p) \oplus k_2(x_{p+1}, \cdots, x_q) \oplus k_3(x_{q+1}, \cdots, x_n).$$

Theorems 2 and 3 are readily generalized to the case where we express f as a product (sum, $\cdots$) of any finite number of other functions.

THE UNIVERSITY OF CALIFORNIA

ON GENERALIZATIONS OF LENGTH AND AREA†

BY J. F. RANDOLPH

1. *Introduction.* In the Lebesgue theory of integration, if G is a bounded Lebesgue measurable point set on the x axis and $f(x)$ is a non-negative bounded Lebesgue measurable function on G, then the plane set H, consisting of all points (x, y) such that x is on G and $0 \leqq y \leqq f(x)$, is Lebesgue plane measurable and $m^{(2)}H = \int_G f(x)dx$. This relation may be proved by showing first that if h is a positive number, then the set H_h: $[x \text{ in } G, 0 \leqq y \leqq h]$ is Lebesgue plane measurable and

$$\text{(A)} \qquad m^{(2)}H_h = h\, mG,$$

that is, the "area" of H_h is its "base" times its "altitude".

C. Carathéodory, W. Gross, and others have defined linear measure for sets not necessarily on a line, plane measure for sets not necessarily on a plane, and in general p-dimensional measure for sets in q-dimensional space, which are generalizations of the notion of curve length, surface area, and so on. Fundamental and simple as it seems, the question whether the generalizations of length and area under these definitions preserve, as do Lebesgue's, the euclidean relation that area is the product of length by length, has received no attention. In this paper we discuss, without answering completely, the simplest phase of this question.

2. *Axioms on General Measure.* We first point out some facts concerning general measure.

By postulating the existence of a set function satisfying five axioms, Carathéodory [2]‡ developed a general theory of measure in which most of the theorems of the usual Lebesgue theory have an analog. Hahn [4], page 444, modified Carathéodory's fifth axiom, and by its use proved also the important relation that the inner measure of a set is the upper limit of the measures of its closed subsets. Hahn's modified axiom is as follows.

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† Presented to the Society, September 12, 1935.

‡ Numbers in brackets refer to the bibliography.

AXIOM V'. *The outer measure of a set is the lower limit of the measures of all containing sets each of which is the intersection of a countable number of open sets.*

3. *Carathéodory's Specific Measures.* Carathéodory also gave the following specific method of attaching a number L^*A to a set A in a euclidean space R_q of q dimensions.

With ρ a positive number let $U_1, U_2, \cdots$ be a sequence of convex point sets open in the space R_q,† each with diameter less than ρ, whose union contains A. With d_k the diameter of U_k consider the sums $d_1+d_2+\cdots$ for all such sequences of point sets. Designate the greatest lower bound, which may be $+\infty$, of such sums by $L_\rho A$. Then $L_\rho A$ does not decrease as ρ decreases. Thus as $\rho \to 0$, $L_\rho A$ approaches a limit, finite or infinite, which in either case is called the outer linear measure of A and is represented by L^*A.

The outer plane measure $L^{*(2)}A$ of a set A is also defined by means of open convex sets $U_1, U_2, \cdots$, each with diameter less than ρ, except that d_k is replaced by the two-dimensional diameter of U_k. The two-dimensional diameter of an open convex set is the least upper bound of the Lebesgue plane measures of the projections of the set on planes of all possible orientations. Arguments similar to those for linear measure show that plane measure also satisfies Hahn's axiom.

4. *Statement of the Question.* The simplest phase of the question under discussion for Carathéodory measure may now be stated as follows.

If A is a plane Carathéodory linearly measurable point set with LA finite, is the cylindrical set J_h: $[(x, y)$ in A, $0 \leqq z \leqq h]$ Carathéodory plane measurable and if so is $L^{(2)}J_h = h\,LA$?

5. *An Inequality for Carathéodory's Measures.* We first prove the following theorem.

THEOREM 1. *If the set A has outer linear measure L^*A finite, then the outer plane measure $L^{*(2)}J_h$ of J_h is less than or equal to hL^*A.*

With ρ a positive number less than unity, let $U_1, U_2, \cdots$

† Carathéodory did not assume the sets U_k to be open, but proved that the same number would be obtained if open convex sets were used.

be a sequence of plane convex point sets whose union contains A and for which $d_k<\rho^2/2$ and $\sum_k d_k<L^*A+\rho$. Now let N_k be the sequence of integers such that $h\leqq N_k(d_k)^{1/2}\leqq h+(d_k)^{1/2}$ and form the three dimensional sets V_{kn} of all points (x, y, z) such that (x, y) is in U_k and $(n-1)(d_k)^{1/2}\leqq z\leqq n(d_k)^{1/2}$, $(n=1, 2, \cdots, N_k)$.

Then each V_{kn} may be included in a right circular cylinder with diameter of base d_k and altitude $(d_k)^{1/2}$. Consequently the diameter of V_{kn} is not greater than $(d_k^2+d_k)^{1/2}<\rho$. Thus, with D_{kn} the two-dimensional diameter of V_{kn}, we have

$$L_\rho^{(2)}(J_h)\leqq\sum_k\sum_n D_{kn}.$$

But the projection on any plane of a right circular cylinder with diameter of base δ and altitude α may be included in a rectangle with dimensions δ and $(\delta^2+\alpha^2)^{1/2}$. Thus $D_{kn}\leqq d_k(d_k^2+d_k)^{1/2}=d_k(d_k)^{1/2}(d_k+1)^{1/2}<(d_k)^{1/2}(\rho+1)^{1/2}d_k$. So $\sum_n D_{kn}\leqq N_k(d_k)^{1/2}\cdot(\rho+1)^{1/2}d_k\leqq(h+(d_k)^{1/2})(\rho+1)^{1/2}d_k<(h+\rho)(\rho+1)^{1/2}d_k$. Consequently $L_\rho^{(2)}(J_h)\leqq(h+\rho)(\rho+1)^{1/2}\sum_k d_k<(h+\rho)(\rho+1)^{1/2}(L^*A+\rho)$. But this is true for each ρ so $L^{*(2)}J_h\leqq hL^*A$. In particular, if A is Carathéodory linearly measurable, then J_h is plane measurable, and

$$\text{(a)}\qquad L^{(2)}J_h\leqq hLA.$$

This follows since any cylindrical set of altitude h with a closed subset K of A as base is a closed subset of J_h. Consequently, the outer plane measures of the part of J_h not in this closed subset is less than or equal to $hL(A-K)$, which may be made arbitrarily small by a proper choice of K.

The question turns then on the reverse inequality

$$\text{(b)}\qquad L^{(2)}J_h\geqq hLA.$$

This inequality, which seems on first thought the more plausible of the two, yields to no method of attack we have been able to devise.

6. *Discussion of Two Measures.* By introducing modifications ΛA and $\Lambda^{(2)}J_h$ of Gross' definitions of measures, we establish an inequality corresponding to (b). The situation is, however, again disconcerting as we are unable to establish for these measures the inequality corresponding to (a).

A comparison of the numbers assigned by the two definitions gives the desired relation between the generalizations of length and area for a certain class of sets. For, as we shall show, $L^{(2)}J_h \geqq \Lambda^{(2)}J_h$ and thus $hLA \geqq L^{(2)}J_h \geqq \Lambda^{(2)}J_h \geqq h\Lambda A$. Consequently, if $LA = \Lambda A$ then $hLA = L^{(2)}J_h = \Lambda^{(2)}J_h = h\Lambda A$. We also obtain conditions on the set A sufficient for the equality $LA = \Lambda A$. Finally we show that our conditions are not necessary for the equality $hLA = L^{(2)}J_h$.

7. *The Gross Outer Measure.* Following Gross, [3], page 156, we fix a rectangular coordinate system and a positive integer k and designate as a partial square the set of all points (x, y) such that $p/2^k \leqq x < (p+1)/2^k$, $q/2^k \leqq y < (q+1)/2^k$, where p and q are integers. There is a countable number of partial squares containing the plane set A which we designate by $w_k^1, w_k^2, \cdots$. Let δ_k^i be the least upper bound of the Lebesgue outer measures of the projections of Aw_k^i on lines of the plane, and form the sums $S_k(A) = \sum_i \delta_k^i$, which may be $+\infty$. Since w_k^i consists of four partial squares $w_{k+1}^{i_1}, \cdots, w_{k+1}^{i_4}$, we have $\delta_k^i \leqq \delta_{k+1}^{i_1} + \cdots + \delta_{k+1}^{i_4}$. Consequently, $S_k(A) \leqq S_{k+1}(A)$, so that $\lim_{k\to\infty} S_k(A)$, which may be $+\infty$, exists. This limit is the Gross outer linear measure of A and is represented by $\Phi_0^*(A)$.

The Gross outer plane measure $\Phi_0^{*(2)}(A)$ of a three-dimensional set A is defined in a similar manner.

This outer linear measure is shown by Gross to satisfy Carathéodory's first four axioms. Furthermore, it is seen that $\Phi_0^*(A)$ is not less than the Lebesgue outer measure of the projection of A on any line. Gross [3], page 158, shows that if $\overline{\Phi}_0^*(A)$ is any other set function satisfying the same four axioms and the projection relation, then $\Phi_0^*(A) \leqq \overline{\Phi}_0^*(A)$; that is, $\Phi_0^*(A)$ is the minimum among such measures and is consequently independent of the coordinate system.

8. *The Outer* Λ *Measures.* The outer linear measure Λ^*A we shall use is the greatest lower bound of the Gross linear measures of all those sets containing A which are the intersections of sequences of open sets. Thus Λ measure satisfies Carathéodory's first four axioms, Hahn's Axiom V', and the projection relation. Consequently, it follows that the inner Λ measure Λ_*A is the upper limit of the Λ linear measures of closed subsets of A.

Furthermore, because Gross' measure is the "smallest" in

the sense used above, Λ measure is the minimum of the measures satisfying Axioms I-V′ and the projection relation. In particular, if K is a closed subset of A, $\Lambda^*(A-K) \leqq L^*(A-K)$. We thus see that if A is Carathéodory linearly measurable with LA finite, then A is also Λ linearly measurable.

The converse of this relation—that if A is Λ linearly measurable, then A is Carathéodory linearly measurable—does not follow. For Besicovitch [1], page 456, and Saks [6] have both constructed closed sets, with positive finite Carathéodory linear measure, whose projection on any line has Lebesgue measure zero. Consequently, any Carathéodory non-linearly measurable subset of either of these sets (if such a subset exists) is still Λ linearly measurable since it has outer Λ linear measure zero.

9. *An Inequality for* Λ *Measures.* We shall now prove the following theorem.

THEOREM 2. *If A is a plane Λ linearly measurable point set with ΛA finite then* $\Lambda^{*(2)} J_h \leqq h\Lambda A$.†

Let $w_k^1, w_k^2, \cdots$ be a sequence of partial squares whose union contains A. Then, with W_k^{ij} the partial cube of all points (x, y, z) such that (x, y) is in w_k^i and $j/2^k \leqq z < (j+1)/2^k$, we see that $\sum_i \sum_j W_k^{ij}$ contains J_h. Then the projection p^i of Aw_k^i on a line of the plane is Lebesgue measurable. For a closed subset K of Aw_k^i projects into a closed subset K_p of p^i and $m^*(p^i - K_p) \leqq \Lambda(Aw_k^i - K)$ and, from the Λ measurability of Aw_k^i, the right side of this inequality may be made arbitrarily small by a proper choice of K. Thus the set P^{ij} of all points (x, y, z) such that (x, y) is in p^i and $j/2^k \leqq z < (j+1)/2^k$ is a Lebesgue plane measurable set which is the projection of $J_h W_k^{ij}$ on a plane parallel to the z axis. Consequently (see equality (A), §1) $m^{(2)} P^{ij} = (1/2^k) m p^i$. Thus with Δ_k^{ij} the least upper bound of the outer Lebesgue plane measures of the projections of $J_h W_k^{ij}$ on planes of all possible orientations, $\Delta_k^{ij} \geqq (1/2^k)\delta_k^i$, and we see that $\Lambda^{*(2)} J_h \geqq h\Lambda A$.

10. *A Sufficient Condition.* We now see that a sufficient condition that J_h be Carathéodory plane measurable and have

† The asterisk is used on the left to indicate outer measure since we are not able to prove here, as in the case of Carathéodory measure, that the measurability of J_h follows from that of A.

$L^{(2)}J_h = hLA$ is that A be Carathéodory linearly measurable with $LA = \Lambda A$. Toward giving a condition under which $LA = \Lambda A$ we make use of the density functions, as defined by Besicovitch [1], page 423. Following him, we shall call a point set A *regular* if A is Carathéodory linearly measurable and the density of A exists and equals unity at almost all points of A.

From a special case of one of Besicovitch's theorems [1], page 426, it follows that at almost all points p of a regular set A there exists a line $l(p)$ through p such that $2r$ and the Lebesgue measure of the projection $[Ac(p, r)]_{l(p)}$ of $Ac(p, r)$ on $l(p)$ are such that

$$\lim_{r \to 0} \frac{m[Ac(p, r)]_{l(p)}}{2r} = 1.$$

Since $m[Ac(p, r)]_{l(p)} \leqq \Lambda Ac(p, r) \leqq LAc(p, r)$, we then see that $\lim_{r \to 0} \Lambda Ac(p, r)/(2r)$ exists and also equals unity at all points p of A except a set of Carathéodory (and consequently Λ) linear measure zero.

Thus, corresponding to an arbitrary positive number ϵ, there exists, for almost all points p of A, a sequence of circles with radii approaching zero, for each circle $c(p, r)$ of which

$$(1 - \epsilon)2r < \Lambda Ac(p, r) \leqq LAc(p, r) < (1 + \epsilon)2r.$$

Let F represent the collection of all such circles. Since the density of A is bounded from zero by unity for almost all of A, we see from [5] Lemma 1, §10, that there exists in the family of circles F a sequence of mutually exclusive circles $c(p, r)$ whose union contains almost all of A. We then observe that

$$LA = \sum_i LAc(p_i, r_i) < (1 + \epsilon) \sum_i 2r_i$$
$$< \frac{(1 + \epsilon) \sum_i \Lambda Ac(p_i, r_i)}{1 - \epsilon} = \frac{(1 + \epsilon)\Lambda A}{1 - \epsilon}.$$

Since ϵ is arbitrary it follows that $LA \leqq \Lambda A$. But $\Lambda A \leqq LA$; hence we have the following result.

THEOREM 3. *If A is a plane regular point set, then*

$$hLA = L^{(2)}J_h = \Lambda^{(2)}J_h = h\Lambda A.$$

11. *The Condition not Necessary.* We now give an example of a

set A which is not regular, but for which $LA = \Lambda A$. This set was constructed by Sierpinski [7], page 184, to show the existence of a set which is not regular.

Let T_0 be an equilateral triangle of side 1. In T_0 form three equilateral triangles of side 1/3, each with a vertex and two sides in common with T_0. Call the union of these three triangles T_1. Now proceed with each triangle of T_1 as was done with T_0, thus obtaining a set T_2 consisting of 3^2 triangles each of side $1/3^2$. In general T_n consists of 3^n equilateral triangles each of side $1/3^n$. The set A is then the intersection $T_0T_1T_2 \cdots$.

Since the projection of A on any side of T_0 completely fills that side, $LA \geqq 1$ and $\Lambda A \geqq 1$. On the other hand, for each n, the point set T_n is a covering of A by 3^n convex point sets each with diameter $1/3^n$. Thus $L_{1/3^n}A \leqq 3^n(1/3^n) = 1$, so $LA \leqq 1$. But $\Lambda A \leqq LA$, so we have $LA = \Lambda A = 1$.

To see that A is not regular, let p be a point of A. Then p belongs to some triangle of T_n for every n. Also the circle $c(p, 1/3^n)$ contains all of one triangle of T_n and not more than one point of any other triangle of T_n. Consequently,

$$\frac{LAc(p, 1/3^n)}{2(1/3^n)} = \frac{1/3^n}{2(1/3^n)} = \frac{1}{2}.$$

We thus see that the lower density of A is less than or equal to 1/2 at each point of A, and A is not regular.

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ON TRANSFORMATIONS OF DOUBLE SERIES

BY H. J. HAMILTON

1. *Introduction and Definition of Notation.* A double series $\sum_{k,l=1}^{\infty} x_{kl}$ may be classified according to the behavior of the double sequence of its partial sums $s_{kl} \equiv \sum_{i,j=1}^{k,l} x_{ij}$ as follows. The sequence $\{s_{kl}\}$ is *ultimately bounded* (abbreviated *ub*) if there exists a number Q such that s_{kl} is bounded for all $k, l > Q$; *bounded* (*b*) if in the preceding case Q can be taken to be zero; *convergent* (*c*) if $\lim_{k,l\to\infty} s_{kl}$ exists (finite); *bounded convergent* (*bc*) if both *b* and *c*; *ultimately regularly convergent* (*urc*) if *c*, and if there exists a number $\bar{Q}$ such that $\lim_{k\to\infty} s_{kl}$ and $\lim_{l\to\infty} s_{kl}$ both exist (finite) for all $l > \bar{Q}$ and all $k > \bar{Q}$, respectively; *regularly convergent* (*rc*) if in the preceding case $\bar{Q}$ can be taken to be zero; *bounded ultimately regularly convergent* (*burc*) if both *b* and *urc*.

It is the purpose of the present paper to establish necessary and sufficient conditions on the matrix $\|b_{kl}\|$ in order that, whenever the series $\sum_{k,l=1}^{\infty} x_{kl}$ is of a specified one of the above types, the transformed series $\sum_{k,l=1}^{\infty} x_{kl} b_{kl}$ will be of a specified one of these types. The process of transforming will be indicated by an arrow; "sufficient" will be abbreviated by *S.*, "necessary" by *N.* Thus $N.b \to c$ reads "a condition (or set of conditions) necessary that every bounded series have a convergent transform," and $S.b \to c$ reads "a condition (or set of conditions) sufficient that every bounded series have a convergent transform."

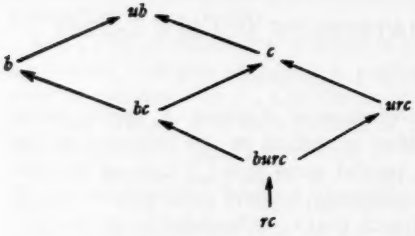
Hardy* found conditions $N.$ and $S.rc \to rc$, and conditions $S.b \to rc$, and established relations (6) and (8) below. Kojima† proved the necessity of Hardy's conditions $S.b \to rc$, and discovered conditions $N.$ and $S.c \to c$. C. N. Moore‡ established conditions $N.$ and $S.bc \to bc$ incidentally, in proving a theorem

* Hardy, *On the convergence of certain multiple series*, Proceedings of the Cambridge Philosophical Society, vol. 19 (1920), pp. 86–95. This paper will be referred to as H.

† Kojima, *Theorems on double series*, Tôhoku Mathematical Journal, vol. 17 (1920), pp. 213–220. This paper will be referred to as K.

‡ C. N. Moore, *On convergence factors in multiple series*, Transactions of this Society, vol. 29 (1927), pp. 227–238. Let $r = 1$, and fix α and β in Moore's Theorem 1.

on Cesàro transforms of double series; the proof of (10) below is based on a suggestion by him.



The seven types of series under consideration are related as in the diagram, where the arrow indicates implication of the quality at its head by that at its tail.

In §2 are listed the conditions to be used in the theorems; §3 establishes various implications of these conditions; §4 is devoted to necessity proofs, and §5 to sufficiency proofs. The results of §§4 and 5 are tabulated in §6, it being noted that if $u \subset U$, and $v \subset V$, where u, U, v, V represent classes of series, then $S.U \to v$ is at once $S.(u, U) \to (v, V)$ (four cases), and $N.u \to V$ is at once $N.(u, U) \to (v, V)$.

Frequent use is made of the following decomposition of double sums, analogous to Abel's partial sum formula for simple series, and due to Hardy.*

$$
\begin{aligned}
\sigma_{mn} \equiv \sum_{k,l=1}^{m,n} x_{kl} b_{kl} &= \sum_{k,l=1}^{m-1,n-1} s_{kl} \Delta_{11} b_{kl} && (\alpha) \\
&+ \sum_{k=1}^{m-1} s_{kn} \Delta_{10} b_{kn} && (\beta) \\
&+ \sum_{l=1}^{n-1} s_{ml} \Delta_{01} b_{ml} && (\gamma) \\
&+ s_{mn} b_{mn}. && (\delta)
\end{aligned} \tag{1}
$$

Here $s_{mn} \equiv \sum_{k,l=1}^{m,n} x_{kl}$; $\Delta_{10} b_{kl} \equiv b_{kl} - b_{k+1,l}$; $\Delta_{01} b_{kl} \equiv b_{kl} - b_{k,l+1}$; and $\Delta_{11} b_{kl} \equiv \Delta_{10}(\Delta_{01} b_{kl}) = \Delta_{01}(\Delta_{10} b_{kl})$. The several parts of the decomposition are lettered for future reference.

2. *Conditions.*

$$
\text{(a)} \quad \sum_{k,l=1}^{\infty} |\Delta_{11} b_{kl}| = A < \infty .
$$

* Hardy, *On the convergence of certain multiple series*, Proceedings of the London Mathematical Society, (2), vol. 1 (1903), pp. 124–128.

(b) $\sum_{k=1}^{\infty} |\Delta_{10} b_{k1}| = B < \infty$; $\sum_{l=1}^{\infty} |\Delta_{01} b_{1l}| = C < \infty$.

(c$_1$) $\lim_{l\to\infty} \Delta_{10} b_{kl} = 0, \quad (k = 1, 2, \cdots)$;
$\lim_{k\to\infty} \Delta_{01} b_{kl} = 0, \quad (l = 1, 2, \cdots)$.

(c$_2$) $\Delta_{10} b_{kl} = 0$ for $l > L_k, \quad (k = 1, 2, \cdots)$;
$\Delta_{01} b_{kl} = 0$ for $k > K_l, \quad (l = 1, 2, \cdots)$.

(d$_1$) $\lim_{l\to\infty} b_{kl} = 0, (k = 1, 2, \cdots)$; $\lim_{k\to\infty} b_{kl} = 0, (l = 1, 2, \cdots)$.

(d$_2$) $b_{kl} = 0$ for $l > J_k, \quad (k = 1, 2, \cdots)$;
$b_{kl} = 0$ for $k > I_l, \quad (l = 1, 2, \cdots)$.

(e) $\lim_{k,l\to\infty} b_{kl} = 0$.

3. *Implications of the Conditions.*

(2) (c$_2$) *implies* (c$_1$).
(3) (d$_1$) *implies* (c$_1$).
(4) (d$_2$) *implies* (c$_2$).
(5) (d$_2$) *implies* (d$_1$).
(6) (a) *and* (b) *imply*

$$\sum_{k=1}^{\infty} |\Delta_{10} b_{kl}| \leqq A + B, \qquad (l = 1, 2, \cdots);$$

$$\sum_{l=1}^{\infty} |\Delta_{01} b_{kl}| \leqq A + C, \qquad (k = 1, 2, \cdots).$$

(7) (a) *and* (b) *imply the existence of*

$\lim_{l\to\infty} \Delta_{10} b_{kl}, \ (k = 1, 2, \cdots)$; $\lim_{k\to\infty} \Delta_{01} b_{kl}, \ (l = 1, 2, \cdots)$.

(8) (a) *and* (b) *imply*

$$\sum_{k=1}^{\infty} \left| \lim_{l\to\infty} \Delta_{10} b_{kl} \right| \leqq A + B; \qquad \sum_{l=1}^{\infty} \left| \lim_{k\to\infty} \Delta_{01} b_{kl} \right| \leqq A + C.$$

(9) (a) *and* (b) *imply that the sequence* $\{b_{kl}\}$ *is rc.*
(10) (a) *and* (c$_1$) *imply* (b).
(11) (a) *and* (c$_1$) *imply*

$$\lim_{l\to\infty} \sum_{k=1}^{\infty} |\Delta_{10} b_{kl}| = 0; \qquad \lim_{k\to\infty} \sum_{l=1}^{\infty} |\Delta_{01} b_{kl}| = 0.$$

(12) (a) *and* (c_1) *imply that*

$$\lim_{l\to\infty} b_{kl} = \lim_{k,l\to\infty} b_{kl}, \qquad (k = 1, 2, \cdots),$$

and that

$$\lim_{k\to\infty} b_{kl} = \lim_{k,l\to\infty} b_{kl}, \qquad (l = 1, 2, \cdots).$$

(13) (a) *and* (d_1) *imply* (e).

(14) (a), (c_1), *and* (e) *imply* (d_1).

Of these implications, the ones which are not obvious can be proved as indicated below. (Where symmetrical results are indicated, only the first is established.)

PROOF OF (6). This follows from the identity, for arbitrary t,

$$\sum_{l=1}^{t-1} \Delta_{11} b_{kl} = \Delta_{10} b_{k1} - \Delta_{10} b_{kt}, \quad (k = 1, 2, \cdots). \tag{15}$$

PROOF OF (7). By (15),

$$\lim_{l\to\infty} \Delta_{10} b_{kl} = \Delta_{10} b_{k1} - \sum_{l=1}^{\infty} \Delta_{11} b_{kl}, \qquad (k = 1, 2, \cdots). \tag{16}$$

PROOF OF (8). This follows from (16).

PROOF OF (9). This follows from the identity, for arbitrary s and t,

$$\begin{aligned}
\sum_{k,l=1}^{s-1,\,t-1} \Delta_{11} b_{kl} &= \sum_{k=1}^{s-1} (\Delta_{10} b_{k1} - \Delta_{10} b_{kt}) \\
&= \sum_{k=1}^{s-1} \Delta_{10} b_{k1} - b_{1t} + b_{st} + b_{11} - b_{11} \\
&= \sum_{k=1}^{s-1} \Delta_{10} b_{k1} + \sum_{l=1}^{t-1} \Delta_{01} b_{1l} - b_{11} + b_{st}.
\end{aligned}$$

PROOF OF (10). Let t be arbitrary. For any s, whatever be $r > t$,

$$\begin{aligned}
\sum_{k=1}^{s} |\Delta_{10} b_{kt}| &\leqq \sum_{k=1}^{s} |\Delta_{10} b_{kt} - \Delta_{10} b_{kr}| + \sum_{k=1}^{s} |\Delta_{10} b_{kr}| \\
&= \sum_{k=1}^{s} \left| \sum_{l=t}^{r-1} \Delta_{11} b_{kl} \right| + \sum_{k=1}^{s} |\Delta_{10} b_{kr}| \\
&\leqq \sum_{k=1,\,l=t}^{\infty} |\Delta_{11} b_{kl}| + \sum_{k=1}^{s} |\Delta_{10} b_{kr}|.
\end{aligned}$$

Hence, by (c_1) and the arbitrariness of s,

$$(17) \qquad \sum_{k=1}^{\infty} |\Delta_{10} b_{kt}| \leqq \sum_{k=1, l=t}^{\infty} |\Delta_{11} b_{kl}|.$$

PROOF OF (11). Let t tend to ∞ in (17).

PROOF OF (12). By (10), (9), and (c_1), the sequence $\{b_{kl}\}$ is *rc*, and

$$\lim_{l\to\infty} b_{kl} = \lim_{l\to\infty} b_{k+1,l}, \qquad (k = 1, 2, \cdots).$$

PROOF OF (13). See (3) and (12).

PROOF OF (14). See (12).

4. *Necessity Proofs.* (Only the first parts of symmetrical conditions are dealt with.)

THEOREM 1. *N.rc→ub are* (a) *and* (b).

The proof for (a) is contained in Theorem 12 of H. That for (b) can be obtained by slight revision of the first argument in Theorem I of K.

THEOREM 2. *N.urc→ub is* (c_2).

By denial of the first part of (c_2), there exists a sequence, for fixed p, of non-zero quantities, $\Delta_{10} b_{pl_i}$, with $l_i > l_{i-1}$. The series with terms

$$x_{kl} = \begin{cases} 1/(\Delta_{10} b_{pl_i}), & (k = p, \quad l = l_i \text{ for } i = 1, 2, \cdots), \\ -1/(\Delta_{10} b_{pl_i}), & (k = p+1, l = l_i \text{ for } i = 1, 2, \cdots), \\ 0, \text{ otherwise}, \end{cases}$$

is *urc*. But, for arbitrary $m > p$ and arbitrary i,

$$\sigma_{ml_i} = \sum_{j=1}^{i} (\Delta_{10} b_{pl_j})/(\Delta_{10} b_{pl_j}) = i.$$

THEOREM 3. *N.urc→b is* (d_2).

By denial of the first part of (d_2), there exists a sequence, for fixed p, of non-zero quantities, b_{pl_i}, with $l_i > l_{i-1}$. The series like that defined in Theorem 2, with $\Delta_{10} b_{pl_i}$ replaced by b_{pl_i}, is *urc*. But for each i,

$$\sigma_{pl_i} = \sum_{j=1}^{i} (b_{pl_i})/(b_{pl_i}) = i.$$

THEOREM 4. *N.burc→c is* (c_1).

Let p be arbitrary. The series with terms

$$x_{kl} = \begin{cases} (-1)^l, & (k = p, \quad l = 1, 2, \cdots), \\ (-1)^{l+1}, & (k = p+1, l = 1, 2, \cdots), \\ 0, \text{ otherwise}, \end{cases}$$

is *burc*. But, for $m > p$ and arbitrary n, $\sigma_{mn} = \sum_{l=1}^{n} (-1)^l \Delta_{10} b_{pl}$, so that $|\Delta_{10} b_{p,n+1}| = |\sigma_{m+1,n+1} - \sigma_{mn}|$, which must tend to zero as m and n tend to ∞.

THEOREM 5. *N.b→c are* (d_1) *and* (e).

The proof for (d_1) is given in Theorem II of K; (e) follows from Theorem 1 and (13) above.

THEOREM 6. *N.bc→urc are* (d_1) *and* (e).

Conditions (a) and (c_1) are *N.bc→urc*, by Theorems 1 and 4, respectively. The series with partial sums $s_{kl} \equiv (-1)^k/l$, (k, $l = 1, 2, \cdots$) is *bc*. In (1), for fixed n, $\lim_{m\to\infty}(\alpha)$ exists, by (a); $\lim_{m\to\infty}(\beta)$ exists, by (10) and (6); $\lim_{m\to\infty}(\gamma) = 0$, by ($c_1$). But by (12) and denial of (e), $\lim_{m\to\infty}(\delta)$ does not exist. Condition (d_1) follows from (14).

THEOREM 7. *N.burc→rc is* (d_1).

Conditions (a) and (c_1) are *N.burc→rc*, by Theorems 1 and 4, respectively. The series with terms

$$x_{kl} = \begin{cases} (-1)^k \quad \text{for } l = 1, & (k = 1, 2, \cdots), \\ (-1)^{k+1} \quad \text{for } l = 2, & (k = 1, 2, \cdots), \\ 0, \text{ otherwise}, \end{cases}$$

is *burc*. But $\sigma_{m1} = \sum_{k=1}^{m} (-1)^k b_{k1}$, so that $|b_{m+1,1}| = |\sigma_{m+1,1} - \sigma_{m1}|$, which must tend to zero as m tends to ∞. Condition (d_1) follows from (12).

5. *Sufficiency Proofs.* [(α), (β), (γ), (δ) are defined in (1).]

THEOREM 8. *S.b→b are* (a) *and* (b).

(α) is bounded, by (a); (β) and (γ) are bounded, by (6); (δ) is bounded, by (9).

THEOREM 9. *S.ub→ub are* (a) *and* (c_2).

Suppose s_{kl} bounded for $k, l>Q$. Set $K\equiv\max K_l$, $(l=1, 2, \cdots, Q)$; $L\equiv\max L_k$, $(k=1, 2, \cdots, Q)$; $R\equiv\max(Q, K, L)$. Since $\Delta_{11}b_{kl}=\Delta_{10}(\Delta_{01}b_{kl})=\Delta_{01}(\Delta_{10}b_{kl})$, (α) is bounded by $\sum_{k,l=1}^{R}|s_{kl}\Delta_{11}b_{kl}|+\sum_{k,l=Q+1}^{\infty}|s_{kl}\Delta_{11}b_{kl}|$. For $n>R$, (β) reduces to $\sum_{k=Q+1}^{m-1}s_{kn}\Delta_{10}b_{kn}$, which is bounded, by (2), (10), and (6). Similarly, for $m>R$, (γ) is bounded. For $m, n>R$, (δ) is bounded by (2), (10), and (9).

THEOREM 10. *S.ub→rc are* (a) *and* (d_2).

The conditions are *S.ub→c*, by (4), (5), (13), and Theorem 15 (below). For fixed n, $\sigma_{mn}=\sum_{l=1,k=1}^{n,I_l}x_{kl}b_{kl}$ for all $m>\max I_l$, $(l=1, 2, \cdots, n)$. Similarly, $\lim_{n\to\infty}\sigma_{mn}$ exists for $m=1, 2, \cdots$.

THEOREM 11. *S.rc→rc are* (a) *and* (b). See Theorem 10 of H.

THEOREM 12. *S.bc→bc are* (a) *and* (c_1).

By (10) and Theorem 8, the conditions are *S.bc→b*. Let m, n tend to ∞. (α) converges, by (a); (β) and (γ) tend to zero, by (11); (δ) converges, by (10) and (9).

THEOREM 13. *S.b→rc are* (a) *and* (d_1).

Use (3), (10), and (13); and see Theorem 11 of H.

THEOREM 14. *S.c→c are* (a) *and* (c_2).

Suppose s_{kl} bounded for $k, l>Q$. Define K, L, R, as in Theorem 9. As m, n tend to ∞, (α) converges to

$$\sum_{k,l=1}^{R}s_{kl}\Delta_{11}b_{kl}+\sum_{k,l=Q+1}^{\infty}s_{kl}\Delta_{11}b_{kl}-\sum_{k,l=Q+1}^{R}s_{kl}\Delta_{11}b_{kl}.$$

For $n>R$, (β) reduces to $\sum_{k=Q+1}^{m-1}s_{kn}\Delta_{10}b_{kn}$, which converges to zero, by (2) and (11). Similarly, it can be shown that (γ) tends to zero. The sum (δ) converges, by (2), (10), and (9). (Compare the concluding argument in Theorem I of K.)

THEOREM 15. *S.ub→urc are* (a), (c_2), *and* (e).

Suppose s_{kl} bounded for $k, l>Q$. Define K, L, R, again as in Theorem 9. As m and n tend to ∞, (α), (β), and (γ) converge, as is seen by reasoning as in Theorem 14; (δ) tends to zero, by (e).

Now fix $n>R$. Then for all $m>\max K_l$, $(l=1, 2, \cdots, n-1)$, (α) becomes $\sum_{l=1,k=1}^{n-1,K_l} s_{kl}\Delta_{11}b_{kl}$; (β) becomes $\sum_{k=Q+1}^{m-1} s_{kn}\Delta_{10}b_{kn}$, and as m tends to ∞ this expression converges, by (2), (10), and (6); (γ) converges to zero, by (2); (δ) tends to zero, by (2) and (14). Similarly, it can be shown that $\lim_{n\to\infty}\sigma_{mn}$ exists for each $m>R$.

THEOREM 16. *S.burc→burc are* (a) *and* (c_1).

The conditions are *S.burc→bc* by Theorem 12. Suppose s_{kl} regularly convergent for $k, l>Q$. For fixed $n>Q$, as m tends to infinity, (α) converges, by (a); (β) converges, by (10) and (6); (γ) converges to zero, by (c_1); (δ) converges, by (10) and (9). Similarly, it can be shown that for fixed $m>Q$, $\lim_{n\to\infty}\sigma_{mn}$ exists.

THEOREM 17. *S.urc→urc are* (a) *and* (c_2).

The conditions are *S.urc→c*, by Theorem 14. Suppose s_{kl} regularly convergent for $k, l>Q$. Define K, L, R, in a manner analogous to that used in Theorem 9. If $n>R$, it can be shown as in Theorem 15 that, as $m\to\infty$, (α), (β), (γ) converge. (δ) converges, by (2), (10), and (9).

6. *Summary of Results.*

Transformation	*N. Conditions*	*S. Conditions*
rc→ ub	(a), (b)	(a), (b)
burc→ ub	(a), (b)	(a), (b)
bc→ ub	(a), (b)	(a), (b)
b→ ub	(a), (b)	(a), (b)
urc→ ub	(a), (b), (c_1), (c_2)	(a), (c_2)
c→ ub	(a), (b), (c_1), (c_2)	(a), (c_2)
ub→ ub	(a), (b), (c_1), (c_2)	(a), (c_2)
rc→ b	(a), (b)	(a), (b)
burc→ b	(a), (b)	(a), (b)
bc→ b	(a), (b)	(a), (b)
b→ b	(a), (b)	(a), (b)
urc→ b	(a), (b), (c_1), (c_2), (d_1), (d_2), (e)	(a), (d_2)
c→ b	(a), (b), (c_1), (c_2), (d_1), (d_2), (e)	(a), (d_2)
ub→ b	(a), (b), (c_1), (c_2), (d_1), (d_2), (e)	(a), (d_2)

Transformation	*N. Conditions*	*S. Conditions*
$rc \rightarrow c$	(a), (b)	(a), (b)
$burc \rightarrow c$	(a), (b), (c_1)	(a), (c_1)
$bc \rightarrow c$	(a), (b), (c_1)	(a), (c_1)
$b \rightarrow c$	(a), (b), (c_1), (d_1), (e)	(a), (d_1)
$urc \rightarrow c$	(a), (b), (c_1), (c_2)	(a), (c_2)
$c \rightarrow c$	(a), (b), (c_1), (c_2)	(a), (c_2)
$ub \rightarrow c$	(a), (b), (c_1), (c_2), (d_1), (e)	(a), (c_2), (e)
$rc \rightarrow bc$	(a), (b)	(a), (b)
$burc \rightarrow bc$	(a), (b), (c_1)	(a), (c_1)
$bc \rightarrow bc$	(a), (b), (c_1)	(a), (c_1)
$b \rightarrow bc$	(a), (b), (c_1), (d_1), (e)	(a), (d_1)
$urc \rightarrow bc$	(a), (b), (c_1), (c_2), (d_1), (d_2), (e)	(a), (d_2)
$c \rightarrow bc$	(a), (b), (c_1), (c_2), (d_1), (d_2), (e)	(a), (d_2)
$ub \rightarrow bc$	(a), (b), (c_1), (c_2), (d_1), (d_2), (e)	(a), (d_2)
$rc \rightarrow urc$	(a), (b)	(a), (b)
$burc \rightarrow urc$	(a), (b), (c_1)	(a), (c_1)
$bc \rightarrow urc$	(a), (b), (c_1), (d_1), (e)	(a), (d_1)
$b \rightarrow urc$	(a), (b), (c_1), (d_1), (e)	(a), (d_1)
$urc \rightarrow urc$	(a), (b), (c_1), (c_2)	(a), (c_2)
$c \rightarrow urc$	(a), (b), (c_1), (c_2), (d_1), (e)	(a), (c_2), (e)
$ub \rightarrow urc$	(a), (b), (c_1), (c_2), (d_1), (e)	(a), (c_2), (e)
$rc \rightarrow burc$	(a), (b)	(a), (b)
$burc \rightarrow burc$	(a), (b), (c_1)	(a), (c_1)
$bc \rightarrow burc$	(a), (b), (c_1), (d_1), (e)	(a), (d_1)
$b \rightarrow burc$	(a), (b), (c_1), (d_1), (e)	(a), (d_1)
$urc \rightarrow burc$	(a), (b), (c_1), (c_2), (d_1), (d_2), (e)	(a), (d_2)
$c \rightarrow burc$	(a), (b), (c_1), (c_2), (d_1), (d_2), (e)	(a), (d_2)
$ub \rightarrow burc$	(a), (b), (c_1), (c_2), (d_1), (d_2), (e)	(a), (d_2)
$rc \rightarrow rc$	(a), (b)	(a), (b)
$burc \rightarrow rc$	(a), (b), (c_1), (d_1), (e)	(a), (d_1)
$bc \rightarrow rc$	(a), (b), (c_1), (d_1), (e)	(a), (d_1)
$b \rightarrow rc$	(a), (b), (c_1), (d_1), (e)	(a), (d_1)
$urc \rightarrow rc$	(a), (b), (c_1), (c_2), (d_1), (d_2), (e)	(a), (d_2)
$c \rightarrow rc$	(a), (b), (c_1), (c_2), (d_1), (d_2), (e)	(a), (d_2)
$ub \rightarrow rc$	(a), (b), (c_1), (c_2), (d_1), (d_2), (e)	(a), (d_2)

Those entries which do not follow from Theorems 1–17 can be deduced from accompanying entries by application of (2), (5), or (13). Such entries are underscored.

BROWN UNIVERSITY

ON CERTAIN TWO-POINT EXPANSIONS OF INTEGRAL FUNCTIONS OF EXPONENTIAL TYPE

BY I. J. SCHOENBERG

1. *Introduction.* Hillel Poritsky* and G. J. Lidstone† found the following formal expansion

$$(1) \qquad f(x) = \sum_{n=0}^{\infty} f^{(2n)}(1)\Lambda_n(x) - \sum_{n=0}^{\infty} f^{(2n)}(0)\Lambda_n(x-1),$$

where $\Lambda_n(x)$ are polynomials (of degree $2n+1$) defined by the generating function

$$\sinh xt \operatorname{cosech} t = \sum_{n=0}^{\infty} t^{2n}\Lambda_n(x).$$

Expansion (1) holds for any polynomial $f(x)$ and solves formally the interpolation problem

$$(2) \qquad f^{(2n)}(1) = a_n, \qquad f^{(2n)}(0) = b_n, \qquad (n \geqq 0).$$

An integral function $f(x)$ is said to be of exponential type if the quantity

$$(3) \qquad \gamma(f) = \overline{\lim_{r\to\infty}} \frac{\log M(r)}{r},$$

which is called the type of the function $f(x)$, is finite, $M(r)$ denoting the maximum modulus of $f(x)$ on the circle $|x| = r$.

Poritsky and J. M. Whittaker‡ proved that the expansion (1)

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* Hillel Poritsky, *On certain polynomial and other approximations to analytic functions*, Transactions of this Society, vol. 34 (1932), pp. 274–331.

† G. J. Lidstone, *Notes on the extension of Aitken's theorem (for polynomial interpolation) to the Everett types*, Proceedings Edinburgh Mathematical Society, vol. 2 (1930), pp. 16–19.

‡ J. M. Whittaker, *On Lidstone's series and two-point expansions of analytic functions*, Proceedings London Mathematical Society, vol. 36 (1933–34), pp. 451–469. In order to facilitate reference we use throughout Whittaker's notations.

is valid in the finite plane for all integral functions of type

$$\gamma(f) < \pi. \tag{4}$$

Whittaker investigated also the interpolation problem

$$f^{(2n)}(1) = a_n, \qquad f^{(2n+1)}(0) = b_n, \qquad (n \geqq 0), \tag{5}$$

which is solved formally by the "generalized Abel expansion"*

$$f(x) = \sum_{n=0}^{\infty} f^{(2n)}(1) M_n(x) - \sum_{n=0}^{\infty} f^{(2n+1)}(0) M'_{n+1}(1-x), \tag{6}$$

where the polynomials $M_n(x)$ (of degree $2n$) are defined by the generating function

$$\cosh xt \operatorname{sech} t = \sum_{n=0}^{\infty} t^n M_n(x).$$

Whittaker showed that the expansion (6) is valid in the finite plane for all integral functions of type

$$\gamma(f) < \frac{\pi}{2}. \tag{7}$$

The consideration of the functions $\sin \pi x$ and $\cos \pi x/2$ shows that π and $\pi/2$ are the best constants on the right sides of the inequalities (4) and (7), respectively, for these functions are respectively solutions of the systems (8) and (10) below.

Our present purpose is to solve the unicity questions connected with the interpolation problems (2) and (5) for functions of exponential type. They are answered completely by the following theorems.†

* See W. Goncharoff, Annales École Normale, vol. 47 (1930), pp. 1–78; also I. J. Schoenberg, *On the zeros of successive derivatives of integral functions*, to appear in the Transactions of this Society.

† It is of interest to mention here a theorem of F. Carlson which, in very restricted form, is as follows: *An integral function $f(x)$ of type $\gamma(f) < \pi$ which satisfies the system*

$$f(n) = 0, \qquad (n = 0, 1, 2, \cdots), \tag{C}$$

must vanish identically (see, for example, E. C. Titchmarsh, *Theory of Functions*, 1932, p. 186). Here there is no analog to Theorem 1 about the solutions of (C) of exponential type, for $\phi(x) \sin \pi x$, with an arbitrary integral function $\phi(x)$, is a solution of (C). Hence such solutions can be of an arbitrary finite type.

THEOREM 1. *The only functions of exponential type satisfying the system*

$$f^{(2n)}(1) = 0, \qquad f^{(2n)}(0) = 0, \qquad (n \geqq 0), \tag{8}$$

are the finite sine polynomials

$$f(x) = \sum_{\nu=1}^{k} c_\nu \sin \nu\pi x, \qquad (\textit{of type } k\pi), \tag{9}$$

with arbitrary constant coefficients c_ν.

THEOREM 2. *The only functions of exponential type satisfying the system*

$$f^{(2n)}(1) = 0, \qquad f^{(2n+1)}(0) = 0, \qquad (n \geqq 0), \tag{10}$$

are the finite cosine polynomials of the form

$$f(x) = \sum_{\nu=0}^{k} c_\nu \cos \frac{(2\nu+1)\pi}{2} x, \qquad (\textit{of type } (2k+1)\pi/2). \tag{11}$$

From Poritsky's and Whittaker's results (4) and (7), it follows only that solutions of (8) and (10) must vanish identically if they are of types less than π and $\pi/2$, respectively.*

2. *Proofs of Theorems* 1 *and* 2. We shall use the following lemma of R. D. Carmichael.†

Let $f(x)$ *be of exponential type not exceeding* γ *and let it have the period* $\omega(>0)$. *Let* $m = [\gamma\omega/2\pi]$ *denote the greatest integer not exceeding* $\gamma\omega/2\pi$. *Then* $f(x)$ *has the form*

$$f(x) = \sum_{\nu=-m}^{m} f_\nu e^{2\pi i\nu x/\omega}, \tag{12}$$

where the f_ν*'s are constants.*

* For a similar extension of a kindred result see A. Weinstein, *Zum Phragmén-Lindelöfschen Ideenkreis*, Abhandlungen Mathematisches Seminar, Hamburg, vol. 6 (1928), pp. 263–264.

† R. D. Carmichael, *Summation of functions of a complex variable*, Annals of Mathematics, (2), vol. 34 (1933), pp. 349–378, Theorem 2.4 on p. 362. A proof, omitted by Carmichael, is readily supplied by means of the Fourier expansion of $f(x)$, $f^{(\omega)}(x)$, their Parseval relations, and Carmichael's Theorem 2.1, p. 361.

Let $f(x)$ be a solution of (8) of finite type; then $f(x)$ *is necessarily an odd function of period* 2. For the equations (8) show that $f(x)$ is odd about the points $x=0$ and $x=1$, whence $f(x)=-f(-x)=f(x+2)$. Applying Carmichael's lemma, for $\omega=2$, we find that $f(x)$ has the form (9), for the cosine terms drop out since $f(x)$ is odd.

To prove Theorem 2 let us assume that $f(x)$ is a solution of (10) of finite type; then $f(x)$ *is necessarily an even function of period* 4. For the equations (10) show that $f(x)$ is even about the point $x=0$ and odd about the point $x=1$, whence $f(x)=f(-x)=-f(x+2)=-f(-x-2)=f(x+4)$. Carmichael's lemma, for $\omega=4$, shows that $f(x)$ is of the form

$$f(x) = \sum_{\nu=0}^{m} d_\nu \cos\frac{\pi\nu}{2}x, \tag{13}$$

the sine terms having dropped out since $f(x)$ is even. Notice that $f(x)$, as defined by (13), always satisfies the second of the equations (10); it also satisfies the first equation (10) provided

$$f^{(2n)}(1) = (-1)^n\pi^{2n}\sum_{j=0}^{[m/2]}(-1)^j d_{2j}\cdot j^{2n} = 0, \tag{14}$$

$$(n = 0, 1, 2, \cdots).$$

The first $[m/2]+1$ equations of the system (14), regarded as a homogeneous system of linear equations in the d_{2j}, suffice already to show that $d_{2j}=0$, $(j=0, 1, \cdots, [m/2])$. Hence (13) reduces to the form (11).

3. *On Whittaker's Solution of the Interpolation Problem* (2). In order to find integral solutions of (2), Whittaker proceeds essentially as follows (loc. cit., Theorem 3, p. 457). Let

$$h(x) = \sum_{n=0}^{\infty}\frac{a_n}{(2n)!}(x-1)^{2n}, \qquad g(x) = \sum_{n=0}^{\infty}\frac{b_n}{(2n)!}x^{2n}, \tag{15}$$

and assume that these are integral functions. Let $H(x)$ and $G(x)$ be integral solutions of the difference equations

$$\begin{aligned} H(x+2) - H(x) &= 2h(x+2), \\ G(x+2) - G(x) &= -2g(x), \end{aligned} \tag{16}$$

respectively. Then

$$(17) \qquad f(x) = \frac{1}{2}\{H(x) - H(-x) + G(x) - G(2-x)\}$$

is an integral function satisfying (1). The proof is immediate, for (15), (16), and (17) imply

$$(18) \qquad f(x) + f(2-x) = 2h(x), \quad f(x) + f(-x) = 2g(x),$$

and these equations are equivalent with the system (2), in view of (15).

If (2) is to have solutions $f(x)$ of exponential type, then $h(x)$ and $g(x)$ must be of such type, as seen from (18). If

$$q = \max\{\gamma(h), \gamma(g)\},$$

it is known that equations (16) have solutions $H(x)$, $G(x)$ of type not exceeding q, and then $f(x)$, as defined by (17), is a solution of (2) of type $\gamma(f) = q$. From Theorem 1 we know that all solutions of (2) which are of exponential type are obtained by adding to $f(x)$ an arbitrary sine polynomial of the form (9).

Carmichael has given very convenient expansions for the "principal sums" $H(x)$ and $G(x)$ (loc. cit., Theorem 2.5, p. 365). Introducing these expansions in formula (17), we get a two-point expansion of $f(x)$ (that is, in terms of $f^{(2n)}(1)$ and $f^{(2n)}(0)$), provided the type q of $f(x)$ is finite, whereas the Poritsky-Lidstone expansion (1) breaks down in general if $q \geqq \pi$. Similar remarks apply to the interpolation problem (5).

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